

# **Applied Mathematical Sciences**

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(continued following index)

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# Regularity Results for Nonlinear Elliptic Systems and Applications



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# Preface

Nonlinear elliptic equations and systems are a classical field of analysis, with many applications in differential geometry, continuum mechanics, and probability theory; an important future branch will be their applications to microelectronics.

The most important analytical tools in the field of nonlinear partial differential equations and systems up to, say, 1955 are presented in the books of C. B. Morrey [79] and O.A. Ladyzhenskaya, N.N. Ural'tseva [66]. The bulk of the development for general nonlinear elliptic systems is presented in M. Giaquinta, E. Giusti [41], D. Gilbarg, N.S. Trudinger [46], later in M. Giaquinta [40]. Concerning applications to differential geometry, we mention the books of M. Giaquinta, S. Hildebrandt [42].

The purpose of this book is to present some of the developments that are not covered in the above books and are promising fields for applications and research.

The book is to a large extent self-contained, with the restriction that the linear theory—Schauder estimates and Campanato theory—is not presented. The reader is expected to be familiar with functional-analytic tools, like the theory of monotone operators. References are given in the text to any techniques that are used. The first two chapters contain general methods and auxiliary lemmas. The expert might like our approach to the theorem of De Giorgi–Nash concerning  $C^\alpha$ -regularity of solutions to nonlinear scalar equations via the hole-filling method, and our proof of Harnack's inequality without using the John–Nirenberg theorem on functions with bounded mean oscillation.

Chapters 4 and 5 deal with diagonal elliptic systems, which have important applications to differential geometry; however, in order to be complementary to the books of Giaquinta–Hildebrandt, we present only the applications to stochastic problems, where the researcher finds challenging open problems with a broad range of degree of difficulty. In fact, the treatment here is more complete than what is available in the literature.

Chapter 6 deals with Helein's proof of the regularity of harmonic mappings on two-dimensional manifolds. We avoid a more extensive study of harmonic mappings, for which we refer to the books of J. Jost [60], M. Giaquinta, S. Hildebrandt [42] (see also J. Eells, J.H. Sampson [22]).

Chapter 7 presents the standard Van Roosbroek equations in semi-conductor theory and a special model that is related to the avalanche effect. We admit that this choice represents a limited sample compared with the range of interesting new open problems waiting to be solved, but in the interest of brevity we have cut the exposition short. In chapter 8 we present recent results for the regularity problem of the Navier–Stokes equation. Clearly, this chapter is not an introduction to mathematical fluid dynamics, for which the reader should refer to the standard book of O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'tseva [67] or, recently, P.L. Lions [72] and G.P. Galdi [36]. We have included this chapter in the book because of surprising similarities of the analytical tools to those in the chapter on diagonal systems. In Chapter 9 we collect results concerning strongly coupled elliptic systems, in particular the theory of A. Koselev [63] concerning sufficient conditions for regularity involving eigenvalues. Chapter 10 presents elements of a dual theory of elliptic systems, the motivation coming from simple models in elasto plasticity. It seems that many techniques in elliptic analysis have a dual analogue. For example, we present a dual proof and formulation of the almost everywhere regularity of solutions of elliptic systems. Chapter 11 contains a short approach to plasticity theory; for the physical background we refer to the books of G. Duvaut, J.L. Lions [15], R. Temam [101] and P. Le Tallec [69]. We believe that the approach via the Norton–Hoff approximation is a recommendable introduction for newcomers who have knowledge of Sobolev spaces. We would like to emphasize that much of the progress concerning the time-dependent Prandtl–Reuss law and regularity properties of its solution has been made by using the dual theory of elliptic equations. This is why it is presented here, although it is a “time”-dependent model, which is in principle outside the scope of this book.

We would like to thank Zamin Iqbal, who carefully read the draft of the book and improved the English to a great extent, and also Josef Malek, who read various parts.

A warm thank you to Chantal Delabarre, who improved the limited LaTeX of the authors, and to Springer-Verlag for publishing this book.

Alain Bensoussan  
Jens Frehse

# Contents

Preface .....	v
<b>1. General Technical Results .....</b>	<b>1</b>
1.1 Introduction .....	1
1.1.1 Function Spaces .....	1
1.1.2 Regularity of Domains .....	10
1.1.3 Poincaré Inequality .....	12
1.1.4 Covering of Domains .....	18
1.2 Useful Techniques .....	25
1.2.1 Reverse Hölder's Inequality .....	25
1.2.2 Gehring's Result .....	36
1.2.3 Hole-Filling Technique of Widman .....	38
1.2.4 Inhomogeneous Hole-Filling .....	40
1.3 Green Function .....	44
1.3.1 Statement of Results .....	44
1.3.2 Proof of Theorem 1.26 .....	45
1.3.3 Estimates on $\log G$ .....	46
1.3.4 Estimates on Positive and Negative Powers of $G$ .....	49
1.3.5 Harnack's Inequality .....	52
1.3.6 Proof of Theorem 1.27 .....	57
<b>2. General Regularity Results .....</b>	<b>63</b>
2.1 Introduction .....	63
2.2 Obtaining $W^{1,p}$ Regularity .....	63
2.2.1 Linear Equations .....	63
2.2.2 Nonlinear Problems .....	66
2.3 Obtaining $C^\delta$ Regularity .....	70
2.3.1 $L^\infty$ Bounds for Linear Problems .....	70
2.3.2 $C^\delta$ Regularity for Dirichlet Problems .....	73
2.3.3 $C^\delta$ Regularity for Linear Mixed Boundary Value Problems .....	82
2.3.4 $C^\delta$ Regularity in the Case $n = 2$ .....	85
2.4 Maximum Principle .....	87
2.4.1 Assumptions .....	87

2.4.2	Proof of Theorem 2.16	88
2.5	More Regularity	89
2.5.1	From $C^\delta$ and $W^{1,p_0}$ , $p_0 > 2$ , to $H_{\text{loc}}^2$	89
2.5.2	Using the Linear Theory of Regularity	96
2.5.3	Full Regularity for a General Quasilinear Scalar Equation	98
<b>3.</b>	<b>Nonlinear Elliptic Systems Arising from Stochastic Games</b>	<b>113</b>
3.1	Stochastic Games Background	113
3.1.1	Statement of the Problem and Results	113
3.1.2	Bellman Equations	115
3.1.3	Verification Property	116
3.2	Introduction to the Analytic Part	118
3.3	Estimates in Sobolev spaces and in $C^\delta$	120
3.3.1	Assumptions and Statement of Results	120
3.3.2	Preliminaries	122
3.3.3	Proof of Theorem 3.7	125
3.4	Estimates in $L^\infty$	127
3.4.1	Assumptions	127
3.4.2	Statement of Results	128
3.5	Existence of Solutions	129
3.5.1	Setting of the Problem and Assumptions	129
3.5.2	Proof of Existence	130
3.5.3	Existence of a Weak Solution	132
3.6	Hamiltonians Arising from Games	133
3.6.1	Notation	133
3.6.2	Verification of the Assumptions for Hölder Regularity	135
3.6.3	Verification of the Assumptions for the $L^\infty$ Bound	136
3.7	The Case of Two Players with Different Coupling Terms in the Payoffs	143
3.7.1	Description of the Model and Statement of Results	144
3.7.2	$L^\infty$ Bounds	145
3.7.3	$H_0^1$ Bound	150
<b>4.</b>	<b>Nonlinear Elliptic Systems Arising from Ergodic Control</b>	<b>153</b>
4.1	Introduction	153
4.2	Assumptions and Statement of Results	154
4.2.1	Assumptions on the Hamiltonians	154
4.2.2	Statement of Results	156
4.3	Proof of Theorem 4.4	156
4.3.1	First Estimates	156
4.3.2	Estimates on $u'_\epsilon - \bar{u}'_\epsilon$	158
4.3.3	End of Proof of Theorem 4.4	161
4.4	Verification of the Assumptions	162
4.4.1	Notation	162



4.4.2	The Scalar Case . . . . .	163
4.4.3	The General Case . . . . .	167
4.5	A Variant of Theorem 4.4 . . . . .	169
4.5.1	Statement of Results . . . . .	169
4.5.2	Proof of Theorem 4.13 . . . . .	170
4.6	Ergodic Problems in $R^n$ . . . . .	175
4.6.1	Presentation of the Problem . . . . .	175
4.6.2	Existence Theorem for an Approximate Solution . . . . .	176
4.6.3	Proof of Theorem 4.17 . . . . .	189
4.6.4	Growth at Infinity . . . . .	191
4.6.5	Uniqueness . . . . .	192
<b>5.</b>	<b>Harmonic Mappings</b> . . . . .	<b>197</b>
5.1	Introduction . . . . .	197
5.2	Extremals . . . . .	198
5.3	Regularity . . . . .	200
5.4	Hardy Spaces . . . . .	201
5.4.1	Basic Properties . . . . .	201
5.4.2	Main Regularity Result in the Hardy Space . . . . .	204
5.5	Proof of Theorem 5.13 . . . . .	208
5.5.1	Continuity when $n = 2$ . . . . .	208
5.5.2	Proof of (5.35) and (5.36) . . . . .	216
5.5.3	Proof of (5.37) . . . . .	218
5.5.4	Atomic decomposition . . . . .	221
<b>6.</b>	<b>Nonlinear Elliptic Systems Arising from the Theory of Semiconductors</b> . . . . .	<b>229</b>
6.1	Physical Background . . . . .	229
6.2	Stationary Case Without Impact Ionization . . . . .	230
6.2.1	Mathematical Setting . . . . .	230
6.2.2	Proof of Theorem 6.1 . . . . .	233
6.2.3	A Uniqueness Result . . . . .	240
6.2.4	Local Regularity . . . . .	245
6.3	Stationary Case with Impact Ionization . . . . .	246
6.3.1	Setting of the Model . . . . .	246
6.3.2	Proof of Theorem 6.5 . . . . .	248
6.4	Impact Ionization Without Recombination . . . . .	257
6.4.1	Statement of the Problem . . . . .	257
6.4.2	Proof of Theorem 6.7 . . . . .	259
<b>7.</b>	<b>Stationary Navier–Stokes Equations</b> . . . . .	<b>265</b>
7.1	Introduction . . . . .	265
7.2	Regularity of “Maximum-Like Solutions” . . . . .	266
7.2.1	Setting of the Problem . . . . .	266

7.2.2	Some Regularity Properties of “Maximum-Like Solutions”	267
7.2.3	The Navier–Stokes Inequality	273
7.2.4	Hole-Filling	275
7.2.5	Full Regularity	279
7.3	Maximum Solutions and the NS Inequality	280
7.3.1	Notation and Setup	280
7.3.2	Proof of Theorem 7.8	281
7.4	Existence of a Regular Solution for $n \leq 5$	283
7.4.1	Green Function Associated with Incompressible Flows	283
7.4.2	Approximation	288
7.4.3	Proof of Existence of a Maximum Solution for $n \leq 5$	289
7.5	Periodic Case: Existence of a Regular Solution for $n < 10$	291
7.5.1	Approximation	291
7.5.2	A Specific Green Function	292
7.5.3	Main Results	295
<b>8.</b>	<b>Strongly Coupled Elliptic Systems</b>	<b>299</b>
8.1	Introduction	299
8.2	$H^2_{\text{loc}}$ and Meyers’s Regularity Results	300
8.3	Hölder Regularity	305
8.3.1	Preliminaries	305
8.3.2	Representation Using Spherical Functions	308
8.3.3	Statement of the Main Result	311
8.3.4	Additional Remarks	317
8.3.5	Hölder’s Continuity up to the Boundary	319
8.4	$C^{1+\alpha}$ Regularity	329
8.4.1	Auxiliary Inequalities	329
8.4.2	Main Result	334
8.5	Almost Everywhere Regularity	338
8.5.1	Regularity on Neighborhoods of Lebesgue Points	338
8.5.2	Proof of Theorem 8.22	339
8.6	Regularity in the Uhlenbeck Case	343
8.6.1	Setting of the Problem	343
8.6.2	Proof of Theorem 8.24	344
8.7	Counterexamples	348
8.8	Regularity for Mixed Boundary Value Systems	352
8.8.1	Stating the Problem	352
8.8.2	Proof of Theorem 8.25	354
8.8.3	Proof of Lemma 8.28	359
8.8.4	Further Regularity	364
8.8.5	Domain with a Corner. Mixed Boundary Conditions	369
8.8.6	Domain with a Corner. Dirichlet Boundary Conditions	371

<b>9. Dual Approach to Nonlinear Elliptic Systems</b> . . . . .	375
9.1 Introduction . . . . .	375
9.2 Preliminaries . . . . .	377
9.2.1 Notation . . . . .	377
9.2.2 Properties of the Operators $\epsilon(u)$ and $Du$ . . . . .	378
9.3 Elasticity Models . . . . .	379
9.3.1 Primal and Dual Problems . . . . .	379
9.3.2 A Hybrid Model . . . . .	380
9.4 $H^1_{loc}$ Theory for the Nonsymmetric Case . . . . .	381
9.4.1 Presentation of the Problem . . . . .	381
9.4.2 $H^1_{loc}$ Regularity . . . . .	382
9.5 $H^1_{loc}$ Theory for the Symmetric Case . . . . .	391
9.5.1 Presentation of the Problem . . . . .	391
9.5.2 $H^1_{loc}$ Regularity . . . . .	391
9.5.3 Reducing the Symmetric Case to the Nonsymmetric Case . . . . .	396
9.6 $L^\infty_{loc}$ Theory for the Nonsymmetric Uhlenbeck Case . . . . .	398
9.6.1 Setting of the Problem and Statement of Results . . . . .	398
9.6.2 Proof of Theorem 9.8 . . . . .	399
9.7 $W^{1,p}_{loc}$ Theory for the Nonsymmetric Case . . . . .	401
9.7.1 Assumptions and Results . . . . .	401
9.7.2 Proof of Theorem 9.9 . . . . .	402
9.8 $C^{1+\delta}_{loc}$ Regularity for the Nonsymmetric Case . . . . .	405
9.8.1 Setting of the Problem and Statement of Results . . . . .	405
9.8.2 Preliminary Results . . . . .	406
9.8.3 Proof of Theorem 9.10 . . . . .	410
9.9 $C^\delta$ Regularity on Neighborhoods of Lebesgue Points for the Nonsymmetric Case . . . . .	413
9.9.1 Setting of the Problem and Statement of Results . . . . .	413
9.9.2 Proof of Theorem 9.11 . . . . .	414
9.9.3 Additional Results in the Uhlenbeck Case . . . . .	418
<b>10. Nonlinear Elliptic Systems Arising from plasticity Theory</b> . . . . .	421
10.1 Introduction . . . . .	421
10.2 Description of Models . . . . .	422
10.2.1 Spaces $U(\Omega), \Sigma(\Omega)$ . . . . .	422
10.2.2 Hencky model . . . . .	423
10.2.3 Norton–Hoff Model . . . . .	424
10.2.4 Passing to the Limit . . . . .	426
10.3 Estimates on the Displacement . . . . .	427
10.3.1 The $f_j$ Derive from a Potential . . . . .	427
10.3.2 Strict Interior Condition . . . . .	428
10.3.3 Constituent Law for the Hencky model . . . . .	429
10.4 $H^1_{loc}$ Regularity . . . . .	430

10.4.1 Preliminaries .....	430
10.4.2 Uniform Estimates and Main Regularity Result .....	432
<b>References</b> .....	<b>435</b>
<b>Index</b> .....	<b>441</b>