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Lectures on Hyperbolic Geometry

With 175 Figures

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A Martina, Paolo, Clara e Stefania
R. B.

Ai miei genitori, Vitta e Luciano
C. P.

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Preface

In recent years hyperbolic geometry was the object and the inspiration for an extensive study which produced important and often amazing results and open questions: it suffices to recall W. P. Thurston's works about the topology and geometry of three-manifolds and the theory of the so-called hyperbolic (or negatively curved) groups. However, it is still difficult to find graduate-level textbooks in the theory of hyperbolic manifolds, starting "from the beginning" and giving a rather complete and reasonably accessible treatise of some recent results. The authors became aware of this difficulty while preparing a course to be held at the Università degli Studi di Pisa, and this book originated from the first notes sketched on that occasion, widely modified and expanded by a profitable collaboration during several months. The aim of this text is to give a modest contribution to the filling of the gap mentioned above, and to the knowledge of such a fascinating field of mathematics.

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One of the main themes of this book is the conflict between the "flexibility" and the "rigidity" properties of the hyperbolic manifolds: the first radical difference arises between the case of dimension 2 and the case of higher dimensions (as proved in chapters B and C), an elementary feature of this phenomenon being the difference between the Riemann mapping theorem and Liouville's theorem, as pointed out in chapter A. This chapter is rather elementary and most of its material may be the object of an undergraduate course.

In chapter B we prove the existence of continuous moduli of hyperbolic structures on a compact surface via a parametrization of the so-called Teichmüller space: we chose the Fenchel-Nielsen parametrization as it is fully placed in the realm of hyperbolic geometry.

Chapter C is devoted to the proof of G. D. Mostow's rigidity theorem (in the compact case). We say two words about this theorem (and the proof we present in the book) in order to point out the difficulties we mentioned above: we chose the Gromov-Thurston proof of the rigidity theorem, as it makes use of a machinery coming essentially from hyperbolic geometry (while Mostow's original proof made extensive use of analysis). However if you follow the chief references [Gro2] and [Th1] a problem arises: the core of the proof (where the differences with Mostow's methods are sharper) consists in establishing a formula relating the Gromov norm of a compact hyperbolic manifold to its volume; following [Th1] and [Gro2] you first consider a larger class of chains than the singular one and then either you extend the norm to this enlarged class or you even change the definition of the norm (and it is not evident that you get a norm equivalent to the original one). The use of these techniques would suit a more advanced course (as it allows one to make the proof shorter and more conceptual) while it is too demanding for a graduate course. We decided to carry out the proof with the usual singular chains and with the natural and elementary definition of the Gromov norm, following a suggestion M. Gromov ascribes to N. H. Kuiper ([Gro2]); we have filled in the necessary details and we have tried to make the proof as transparent as possible.

Together with the rigidity theorem, a basic tool for the study of hyperbolic manifolds is Margulis' lemma, a detailed proof of which we give in chapter D; as a consequence of this result in the same chapter we also give a rather accurate description, in all dimensions, of the thin-thick decomposition of a hyperbolic manifold (especially in case of finite volume).

Chapter E is devoted to the space of hyperbolic manifolds and to the volume function. We start with the introduction of a natural topology (the so-called geometric topology) on the space of all hyperbolic manifolds having fixed dimension $n \geq 2$, and we discuss different characterizations of such a topology (we shall be interested in particular in the notion of convergence of a sequence). As a corollary of this discussion we shall obtain quite easily the fact that for $n \geq 3$ the volume function (defined on the space of finite-volume hyperbolic n-manifolds) is continuous and proper. This result, together with an extensive use of the fundamental tools developed in the previous chapters (the rigidity theorem and the study of the thin-thick decomposition), will allow us to prove Wang's theorem for hyperbolic manifolds of dimension $n \geq 4$ and most of the so-called Jorgensen-Thurston theory for $n = 3$ (the case of dimension 2 will be treated independently.) The radical difference of behaviour between the case $n \geq 4$ and the case $n = 3$ can be seen as another very important example of the conflict rigidity-flexibility we mentioned above. We shall point out the way the exception of the case $n = 3$ depends essentially on the purely topological fact that all closed 3-manifolds can be obtained via Dehn surgery along links (for instance, in S^3), and on the crucial remark that "almost all" these surgeries can be "made hyperbolic": this is Thurston's hyperbolic Dehn surgery theorem, of which we give a detailed proof based on the possibility of expressing a non-compact hyperbolic three-manifold as ideal tetrahedra with glued faces; this proof is as far as possible elementary and constructive.

Though we confine ourselves to the major aspects of this theory, we are confident that the results explicitly proved in this book are enough to appreciate the sharp difference between the cases $n = 3$ and $n \geq 4$. On the other hand Jorgensen-Thurston theory provides some information whose proof requires the extension of the definition of the Gromov norm and of the techniques used in the proof of the rigidity theorem: we shall give sketches of these results and quote the references for the proofs. We shall also mention other features of flexibility in dimension 3.

We want to emphasize that our discussion of the volume function is mostly deduced from the properties of the natural topology (whose existence is proved *a priori*) on the set of all hyperbolic n-manifolds. As for Jorgensen-Thurston theory the line we shall follow presents some remarkable differences from the chief reference [Th1, ch. 5,6]: a reason is that in these notes we met a difficulty we were not able to overcome, so we needed to re-organize many proofs (we refer to section E.4 for a discussion of these facts).

The notion of Gromov norm introduced in chapter C for the proof of the rigidity theorem can be naturally placed in the general theory of bounded co-homology (developed in [Gro3]); indeed, we can say that proportionality between the Gromov norm of the fundamental class and the volume of a hyperbolic manifold provides the first natural example showing that the theory of bounded co-homology is non-trivial. In chapter F, very far from being complete, we shall just briefly sketch a few other viewpoints of this theory and try to provide some more motivations for it. In particular we shall discuss another interesting example of non-trivial bounded class coming from the study of the Euler class of a flat fiber bundle (due to Milnor, Wood, Sullivan, Gromov and others). In this context we will meet the notion of amenable group, to which we shall devote some space.

The list of references has no pretensions of completeness: it represents the texts we actually used during our work.

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While drawing up this text we have tried to be as self-contained as possible, though we are conscious that this aim remains often closer to an aspiration than to an actual realization: for instance the knowledge of the very basic notions of Riemannian geometry and algebraic topology is essential for a complete understanding of the most part of the book. On the other hand almost all the results mentioned are explicitly proved, and for those which are not easily accessible bibliographical references are given. We hope our aim to be self-contained has been realized at least in the following weak form: the reader can follow the topics and the techniques of this text without needing to stop too often and fill some gap in the pre-requisites, and, in the meanwhile, without feeling he is being asked to accept too many acts of faith.

Finally, a few acknowledgements: it is quite evident from the present preface that this text was largely influenced by the works of W. P. Thurston and M. Gromov; indeed we could say its aim is to divulge in an accessible way a (very little) part of their work. Personally, we are keen on saying that the present text owes much to M. Boileau, G. Levitt, J. C. Sikovav and to the course they organized at Orsay in 1987/88: the first author had the good fortune to attend it, and this fact certainly influenced the choice of the topics and sometimes the details of the proofs. This is true in particular for the section concerning amenable groups, which is largely inspired by some notes Sikovav wrote on that occasion. The second author would like to thank the University of Warwick; he was a visitor there when the final version of the book was completed and he used the University's computer facilities extensively. We also acknowledge some valuable suggestions concerning chapter E made by C. C. Adams. Lastly, we warmly thank Andrea Petronio for the very accurate illustrations and David Trotman for his help in checking our English.

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