

Gerd Baumann

Symmetry Analysis of  
Differential Equations  
with *Mathematica*<sup>®</sup>



Springer Science+Business Media, LLC

Gerd Baumann  
Department of Mathematical Physics  
University of Ulm  
Ulm D-89069  
Germany

Library of Congress Cataloging-in-Publication Data  
Baumann, Gerd.

Symmetry analysis of differential equations with Mathematica /

Gerd Baumann.

p. cm.

Includes bibliographical references and indexes.

**Additional material to this book can be downloaded from <http://extras.springer.com>.**

ISBN 978-1-4612-7418-6

ISBN 978-1-4612-2110-4 (eBook)

DOI 10.1007/978-1-4612-2110-4

1. Differential equations—Numerical solutions—Computer programs.
2. Symmetry (Physics) 3. Mathematica (Computer program language)

I. Title

QA371.B36 1998

515'.35—dc21

98-26975

Printed on acid-free paper.

© 2000 Springer Science+Business Media New York

Originally published by Springer-Verlag New York, Inc. 2000

Softcover reprint of the hardcover 1st edition 2000

TELOS®, The Electronic Library of Science, is an imprint of Springer-Verlag New York, Inc.

This Work consists of a printed book and a CD-ROM packaged with the book, both of which are protected by federal copyright law and international treaty. The book may not be translated or copied in whole or in part without the written permission of the publisher (Springer Science+Business Media, LLC), except for brief excerpts in connection with reviews or scholarly analysis. For copyright information regarding the CD-ROM, please consult the printed information packaged with the CD-ROM in the back of this publication, and which is also stored as a “readme” file on the CD-ROM. Use of the printed version of this Work in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known, or hereafter developed, other than those uses expressly granted in the CD-ROM copyright notice and disclaimer information, is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone. Where those designations appear in the book and Springer-Verlag was aware of a trademark claim, the designations follow the capitalization style used by the manufacturer.

Production managed by Steven Pisano; manufacturing supervised by Jacqui Ashri.

Typeset by Integre Technical Publishing Co., Inc., Albuquerque, NM.

9 8 7 6 5 4 3 2 1

---

# Contents

	<b>Preface</b>	<b>vii</b>
<b>Chapter 1</b>	<b>Introduction</b>	<b>1</b>
<b>Chapter 2</b>	<b>Elements of Symmetry Analysis</b>	<b>6</b>
	2.1 Groups and Lie Groups	6
	2.1.1 Groups	6
	2.1.2 Isomorphism	14
	2.1.3 Lie Groups	14
	2.2 Lie Algebras	21
	2.2.1 Representation of a Lie Algebra	26
	2.2.2 Properties of Lie Algebras	29
<b>Chapter 3</b>	<b>Derivatives</b>	<b>37</b>
	3.1 Ordinary and Partial Derivatives	37
	3.2 Tangent Vector	45
	3.3 The Total Derivative	50
	3.4 Prolongations	52
	3.5 The Fréchet Derivative	54
	3.6 The Euler Derivative	59
	3.6.1 The Problem of Variation	59
	3.6.2 Euler's Equation	63
	3.6.3 Euler Operator	65
	3.6.4 Algorithm Used in the Calculus of Variations	65
	3.6.5 Euler Operator for $q$ Dependent Variables	69

3.6.6 Euler Operator for  $q + p$  Dimensions 71  
 3.7 Prolongation of Vector Fields 74

**Chapter 4 Symmetries of Ordinary Differential Equations 96**

4.1 Introduction 96  
 4.2 Symmetry Transformations of Functions 98  
     4.2.1 Symmetries 98  
     4.2.2 Infinitesimal Transformations 103  
     4.2.3 Group Invariants 107  
     4.2.4 Tangent Vector 112  
     4.2.5 Prolongation of Transformations 117  
 4.3 Symmetry Transformations of Differential Equations 123  
     4.3.1 Definition of a Symmetry Group 123  
     4.3.2 Main Properties of Symmetry Groups 124  
     4.3.3 Calculation of the Infinitesimal Symmetries 125  
     4.3.4 Canonical Variables 139  
 4.4 Analysis of Ordinary Differential Equations 148  
     4.4.1 First-Order Equations 148  
     4.4.2 Second-Order Ordinary Differential Equations 174  
     4.4.3 Higher-Order Ordinary Differential Equations 201

**Chapter 5 Point Symmetries of Partial Differential Equations 216**

5.1 Introduction 216  
 5.2 Lie's Theory Used in MathLie 217  
 5.3 Invariance Based on Fréchet Derivatives 220  
 5.4 Application of the Theory 222  
     5.4.1 Calculation of Prolongations 223  
     5.4.2 Derivation of Determining Equations 229  
     5.4.3 Interactive Solution of Determining Equations 235  
     5.4.4 Data Basis of Symmetries 243  
 5.5 Similarity Reduction of Partial Differential Equations 257  
 5.6 Working Examples 282  
     5.6.1 The Diffusion Equation 282  
     5.6.2 The Earthworm's New Year Problem 282  
     5.6.3 Single Flux Line in Superconductors 289  
     5.6.4 The Korteweg-de Vries Equation and its Generalizations 296  
     5.6.5 Stokes' Solution of the Creeping Flow 304  
     5.6.6 Two-Dimensional Boundary-Layer Flows: Group Classification 311  
     5.6.7 The Plane Jet 323  
     5.6.8 Drop Formation 330  
     5.6.9 The Rayleigh Particle 340  
     5.6.10 Molecular Beam Epitaxy 346  
     5.6.11 The First Atomic Explosion 355

<b>Chapter 6</b>	<b>Non-Classical Symmetries of Partial Differential Equations</b>	<b>365</b>
	6.1 Introduction	365
	6.2 Mathematical Background of the Non-classical Method	366
	6.3 Applications of the Non-classical Method	370
	6.3.1 The Heat Equation	370
	6.3.2 The Boussinesq Equation	377
	6.3.3 The Fokker-Planck Equation	383
<b>Chapter 7</b>	<b>Potential Symmetries of Partial Differential Equations</b>	<b>392</b>
	7.1 Introduction	392
	7.2 Basics of Potential Symmetries	393
	7.3 Calculation of Potential Symmetries	394
	7.4 Applications of Potential Symmetries	398
	7.4.1 A Non-linear Reaction Diffusion Equation	398
	7.4.2 Cylindrical Korteweg-de Vries Equation	399
	7.4.3 The Burgers Equation	402
<b>Chapter 8</b>	<b>Approximate Symmetries of Partial Differential Equations</b>	<b>404</b>
	8.1 Introduction	404
	8.2 Approximations	405
	8.3 One-Parameter Approximation Group	405
	8.4 Approximate Group Generator	407
	8.5 The Determining Equations and an Algorithm of Calculation	408
	8.6 Examples	410
	8.6.1 Isentropic Liquid	410
	8.6.2 Perturbed Korteweg-de Vries Equation	419
<b>Chapter 9</b>	<b>Generalized Symmetries</b>	<b>424</b>
	9.1 Introduction	424
	9.2 Elements of Generalized Symmetries	425
	9.3 Algorithm for Calculation of Generalized Symmetries	427
	9.4 Examples	428
	9.4.1 Diffusion Equation	428
	9.4.2 Potential Burgers Equation	430
	9.4.3 Generalized Korteweg-de Vries Equations	431
	9.4.4 Coupled System of Wave Equations	432
	9.5 Second-Order ODEs and the Euler-Lagrange Equation	433
	9.5.1 Generalized Symmetries and Second-Order ODEs	434
	9.5.2 Conservation Laws	436
	9.6 Algorithm for Conservation Laws of Second-Order ODEs	437
	9.7 Examples for Second-Order ODEs	438
	9.7.1 The Hénon-Heiles Model	438
	9.7.2 Two-Dimensional Quartic Oscillators	446
	9.7.3 Two Ions in a Trap	452

<b>Chapter 10</b>	<b>Solution of Coupled Linear Partial Differential Equations</b>	<b>457</b>
10.1	Introduction	457
10.2	General Canonical Form of PDEs	458
	10.2.1 Application of the General Canonical Form Algorithm	462
10.3	Solution of Linear PDEs	471
	10.3.1 Integration of Monomials	472
	10.3.2 Integrating ODEs and Pseudo-ODEs	473
	10.3.3 Integrating Exact PDEs	473
	10.3.4 Potential Representation	474
10.4	Simplification of Equations	475
	10.4.1 Direct Separation	475
	10.4.2 Indirect Separation	476
	10.4.3 Reducing the Number of Dependent Variables	477
10.5	Example	479
	10.5.1 Liouville-Type Equation of Quantum Gravity Theory	480
<b>Chapter 11</b>	<b>Appendix</b>	<b>483</b>
	A Marius Sophus Lie: A Mathematician's Life	483
	B List of Key Symbols Used in <i>Mathematica</i>	487
	C Installing <i>MathLie</i>	488
	<b>References</b>	<b>493</b>
	<b>Index for <i>MathLie</i> and <i>Mathematica</i> Functions</b>	<b>503</b>
	<b>Subject Index</b>	<b>505</b>