

Lecture Notes in Mathematics

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Topological Methods for Variational Problems with Symmetries

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Preface

Symmetry has a strong impact on the number and shape of solutions to variational problems. This can be observed, for instance, when one looks for periodic solutions of autonomous differential equations and exploits the invariance under time shifts or when one is interested in elliptic equations on symmetric domains and wants to find special solutions.

Two topological methods have been devised in order to find critical points that are not minima or maxima of variational integrals: the theories of Lusternik-Schnirelmann and of Morse. In these notes we want to present recent techniques and results in critical point theory for functionals invariant under a symmetry group. We develop Lusternik-Schnirelmann (minimax) theory and a generalization of Morse theory, the Morse-Conley theory, in some detail. Both theories are based on topological notions: the Lusternik-Schnirelmann category and geometrical index theories like the genus on the one hand and the Conley index (generalizing the Morse index) on the other hand. These notions belong to the realm of homotopy theory with a typical consequence: They are easy to define but difficult to compute.

We present a variety of new computations of the category where very general classes of symmetry groups are involved, and we give examples showing that our results cannot be extended further without serious restrictions. In order to do this we prove new generalizations of the Borsuk-Ulam theorem and give counterexamples to more general versions. It is here that we need to use some algebraic topology, namely cohomology theory, and an equivariant version of the cup-length.

A variation of the equivariant cup-length, the “length”, turns out to be very useful also in our presentation of the Morse-Conley theory for symmetric flows. The length is a cohomological index theory. It should be considered as a measure for the size of invariant sets. We use it to associate an integer, the “exit-length”, to an isolated invariant subset of a flow. The exit-length is closely related to the Morse index of a nondegenerate critical point. We apply our general theory to bifurcation problems and show that a change of the exit-length along a branch of stationary solutions gives rise to bifurcation. The length of the bifurcating set of bounded solutions can be estimated, and this in turn can be used to analyze the bifurcating set. This approach to bifurcation theory will be illustrated with two applications, namely the bifurcation of steady states and heteroclinic orbits of $O(3)$ -symmetric flows, and with the existence of periodic solutions near equilibria of symmetric Hamiltonian systems. We discuss the symmetry of the bifurcating solutions, obtain multiplicity results and discover special solutions. For example, we find brake orbits and normal mode solutions for certain symmetric Hamiltonian systems.

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Contents

Preface	VII
Contents	IX
Chapter 1. Introduction	
1.1 The question	1
1.2 The approach	4
1.3 An advice	7
Chapter 2. Category, genus and critical point theory with symmetries	
2.1 Introduction	8
2.2 Equivariant topology	9
2.3 Category and genus	11
2.4 Properties of category and genus	15
2.5 Critical point theory with symmetries	20
2.6 A symmetric mountain pass theorem	24
Chapter 3. Category and genus of infinite-dimensional representation spheres	
3.1 Introduction	30
3.2 Statement of results	31
3.3 G-maps between spheres	34
3.4 Proofs	41
3.5 Related results and examples	46
Chapter 4. The length of G-spaces	
4.1 Introduction	53
4.2 Equivariant cohomology theories	54
4.3 The length	57
4.4 Properties of the length	63
4.5 More properties for special groups	67
Chapter 5. The length of representation spheres	
5.1 Introduction	72
5.2 Torus groups	73
5.3 Cyclic p-groups	76
5.4 Proof of Theorem 3.1a)	80
5.5 Related results	82

Chapter 6. The length and Conley index theory

6.1 Introduction	86
6.2 The equivariant Conley index	87
6.3 Stationary solutions and connecting orbits	89
6.4 Hyperbolicity and continuation	93

Chapter 7. The exit-length

7.1 Introduction	96
7.2 The exit-length of isolated invariant sets	97
7.3 Continuation of the exit-length	99
7.4 Properties of the exit-length	101
7.5 A bifurcation theorem	105

Chapter 8. Bifurcation for $O(3)$ -equivariant problems

8.1 Introduction	113
8.2 Subgroups and representations of $O(3)$	114
8.3 Bifurcating stationary solutions	117
8.4 Bifurcating connecting orbits	120

Chapter 9. Multiple periodic solutions near equilibria of symmetric Hamiltonian systems

9.1 Introduction	127
9.2 Fixed energy	128
9.3 Fixed period	133
9.4 Symmetric periodic solutions	136

References	142
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Index	150
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