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Topological Methods for Variational Problems with Symmetries

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Preface

Symmetry has a strong impact on the number and shape of solutions to variational problems. This can be observed, for instance, when one looks for periodic solutions of autonomous differential equations and exploits the invariance under time shifts or when one is interested in elliptic equations on symmetric domains and wants to find special solutions.

Two topological methods have been devised in order to find critical points that are not minima or maxima of variational integrals: the theories of Lusternik-Schnirelmann and of Morse. In these notes we want to present recent techniques and results in critical point theory for functionals invariant under a symmetry group. We develop Lusternik-Schnirelmann (minimax) theory and a generalization of Morse theory, the Morse-Conley theory, in some detail. Both theories are based on topological notions: the Lusternik-Schnirelmann category and geometrical index theories like the genus on the one hand and the Conley index (generalizing the Morse index) on the other hand. These notions belong to the realm of homotopy theory with a typical consequence: They are easy to define but difficult to compute.

We present a variety of new computations of the category where very general classes of symmetry groups are involved, and we give examples showing that our results cannot be extended further without serious restrictions. In order to do this we prove new generalizations of the Borsuk-Ulam theorem and give counterexamples to more general versions. It is here that we need to use some algebraic topology, namely cohomology theory, and an equivariant version of the cup-length.

A variation of the equivariant cup-length, the "length", turns out to be very useful also in our presentation of the Morse-Conley theory for symmetric flows. The length is a cohomological index theory. It should be considered as a measure for the size of invariant sets. We use it to associate an integer, the "exit-length", to an isolated invariant subset of a flow. The exit-length is closely related to the Morse index of a nondegenerate critical point. We apply our general theory to bifurcation problems and show that a change of the exit-length along a branch of stationary solutions gives rise to bifurcation. The length of the bifurcating set of bounded solutions can be estimated, and this in turn can be used to analyze the bifurcating set. This approach to bifurcation theory will be illustrated with two applications, namely the bifurcation of steady states and heteroclinic orbits of $O(3)$-symmetric flows, and with the existence of periodic solutions near equilibria of symmetric Hamiltonian systems. We discuss the symmetry of the bifurcating solutions, obtain multiplicity results and discover special solutions. For example, we find brake orbits and normal mode solutions for certain symmetric Hamiltonian systems.

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