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(continued after index)

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An Introduction to Riemann–Finsler Geometry

With 20 Illustrations

Springer

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To Our Teachers
in Life and in Mathematics

Preface

A historical perspective

The subject matter of this book had its genesis in Riemann's 1854 "habilitation" address: "Uber die Hypothesen, welche der Geometrie zu Grunde liegen" (On the Hypotheses, which lie at the Foundations of Geometry). Volume II of Spivak's *Differential Geometry* contains an English translation of this influential lecture, with a commentary by Spivak himself.

Riemann, undoubtedly the greatest mathematician of the 19th century, aimed at introducing the notion of a manifold and its structures. The problem involved great difficulties. But, with hypotheses on the smoothness of the functions in question, the issues can be settled satisfactorily and there is now a complete treatment.

Traditionally, the structure being focused on is the Riemannian metric, which is a quadratic differential form. Put another way, it is a smoothly varying family of inner products, one on each tangent space. The resulting geometry — Riemannian geometry — has undergone tremendous development in this century. Areas in which it has had significant impact include Einstein's theory of general relativity, and global differential geometry.

In the context of Riemann's lecture, this restriction to a quadratic differential form constitutes only a special case. Nevertheless, Riemann saw the great merit of this special case, so much so that he introduced for it the curvature tensor and the notion of sectional curvature. Such was done through a Taylor expansion of the Riemannian metric.

The Riemann curvature tensor plays a major role in a fundamental problem. Namely: how does one decide, in principle, whether two given Riemannian structures differ only by a coordinate transformation? This was solved in 1870, independently by Christoffel and Lipschitz, using different methods and *without* the benefit of tensor calculus. It was almost 50 years later, in 1917, that Levi-Civita introduced his notion of parallelism (equivalent to a connection), thereby giving the solution a simple geometrical interpretation.

Riemann saw the difference between the quadratic case and the general case. However, the latter had no choice but to lay dormant when he remarked that "The study of the metric which is the fourth root of a quartic differential form is quite time-consuming and does not throw new light to the problem." Happily, interest in the general case was revived in 1918 by Paul Finsler's thesis, written under the direction of Carathéodory. For this reason, we refer to the general case as Riemann–Finsler geometry, or Finsler geometry for short.

Finsler geometry is closely related to the calculus of variations. See §1.0. As such its deeper study went back at least to Jacobi and Adolf Kneser. In his Paris address in 1900, Hilbert formulated 23 unsolved problems. The last one was devoted to the geometry of the calculus of variations. It is the only problem for which he did not formulate a specific question/conjecture. Hilbert gave praise to Kneser's book, then new. He provided an account of the invariant integral, and emphasized the importance of the problem of multiple integrals. The Hilbert invariant integral plays an important role in all modern treatments of the subject.

The geometrical data in Finsler geometry consists of a smoothly varying family of Minkowski norms (one on each tangent space), rather than a family of inner products. This family of Minkowski norms is known as a Finsler structure. Just like Riemannian geometry, there is the equivalence problem: how can one decide (in principle) whether two given Finsler structures differ only by a transformation induced from a coordinate change? It is not unreasonable to expect that the solution of the equivalence problem will again involve a connection and its curvature, together with the proper space on which these objects live.

In Riemannian geometry, the connection of choice was that constructed by Levi-Civita, using the Christoffel symbols. It has two remarkable attributes: metric-compatibility and torsion-freeness. Although we now know that in Finsler geometry *proper*, these cannot both be present in the same connection, such was perhaps not common knowledge during the turn of the century. Even after reaching this realization, one still faces the daunting task of writing down viable structural equations for the connection. Furthermore, the Levi-Civita (Christoffel) connection operates on the tangent bundle TM of our underlying manifold M. But the same cannot be said of its Finslerian counterpart.

It was not until 1926 that significant progress was made by Ludwig Berwald (1883–1942), from an analytical perspective. See the poignant and informative obituary by Max Pinl in Scripta Math. **27** (1965), 193–203.

Berwald's work stemmed from the study of systems of differential equations, and was very much rooted in the calculus of variations. He introduced a connection and two curvature tensors, all rightfully bearing his name. See Matsumoto's appendix ("A History of Finsler Geometry") in *Proceedings of the 33rd Symposium on Finsler Geometry* (ed. Okubo), 1998, Lake Yamanaka. (A revised version is scheduled to appear in Tensor.) The Berwald connection is torsion-free, but is (necessarily) not metric-compatible. The Berwald curvature tensors are of two types: an *hh*- one not unlike the Riemann curvature tensor, and an *hv*- one which automatically vanishes in the Riemannian setting. Berwald's constructions have, since their inception, been indispensable to the geometry of path spaces.

Enthusiasts of metric-compatibility were not to be outdone. It is an amusing irony that although Finsler geometry starts with only a norm in any given tangent space, it regains an entire family (!) of inner products, one for each direction in that tangent space. This is why one can still make sense of metric-compatibility in the Finsler setting. In 1934, Elie Cartan introduced a connection that is metric-compatible but has torsion. The Cartan connection remains, to this day, immensely popular with the Matsumoto and the Miron schools of Finsler geometry. Besides the curvature tensors of *hh*- and *hv*- type, there is a third curvature tensor associated with the Cartan connection. It is of *vv*- type. Curiously, this last tensor is numerically identical to the curvature of a canonical (albeit singular) Riemannian metric on each tangent space.

Back in the torsion-free camp, the next progress came in 1948, when the Chern connection was discovered. Its formula differs from that of Berwald's by an $\dot{A}$ term. In natural coordinates on the slit tangent bundle $TM \setminus 0$, the Chern connection coefficients are given by

$$\frac{g^{is}}{2}\left(\frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j}\right) .$$

To get those for the Berwald connection, one simply adds on the tensor $\dot{A}^i{}_{jk}$. More importantly, replacing the operator $\frac{\delta}{\delta x}$ by $\frac{\partial}{\partial x}$ gives the familiar Levi-Civita (Christoffel) connection of Riemannian metrics.

The connections of Berwald and Chern are both torsion-free. They also fail, slightly but expectedly, to be metric-compatible. Of the two, the Chern connection is simpler in form, while the Berwald connection effects a leaner *hh*-curvature for spaces of constant flag curvature. These connections coincide when the underlying Finsler structure is of Landsberg type. They further reduce to a *linear* connection on M, one which operates on TM, when the Finsler structure is of Berwald type.

In the generic Finslerian case, none of the connections we mentioned operates directly on the tangent bundle TM over M. Chern realized in his solution of the equivalence problem that, by pulling back TM so that it sits over the manifold of rays SM rather than M, one provides a natural vector bundle on which these connections may operate. It is within this geometrized setting that the equivalence problem and its solution admit a sound conceptual interpretation.

The layout of the book

The Riemann–Finsler manifolds form a much larger class than the Riemannian manifolds. Correspondingly, the former has a much more extensive literature, connected with the names Synge, Berwald, E. Cartan, Busemann, Rund, and many of our contemporaries. It is *not* the objective of this book to provide a comprehensive survey. Rather, following the general outline of Riemann and Hilbert, our aim is to develop the subject somewhat independently, with Riemannian geometry as a special case. We hope our attempt at least reflects some of the spirits of those two pioneers.

This book is comprised of three parts:

* Finsler Manifolds and Their Curvature: four chapters.
* Calculus of Variations and Comparison Theorems: five chapters.
* Special Finsler Spaces over the Reals: five chapters.

The key points of each chapter are detailed in our table of contents. Given that, we refrain from discussing here the specific topics covered.

There are fourteen chapters with an average of 30 pages each. The chapters are intentionally kept short. It seems that psychologically, one's progress through the Finsler landscape is more easily monitored this way. Every chapter is devoted to (only) one or two major results. This constraint allows us to base each chapter on a *single* theme, thereby rendering the book more teachable.

Regarding classroom use, the students we have in mind are advanced undergraduates or first-year graduate students. They are assumed to have had at least a small amount of tensor analysis, to the extent that they are comfortable with the gymnastics of raising and lowering indices. It would also help if they have had some exposure to manifolds in the abstract, so that pull-backs and push-forwards are familiar operations. Some computational experience with the Gaussian curvature of Riemannian surfaces would provide adequate motivation and intuition. This book contains enough material for roughly three semester courses.

We have adopted a candid style of writing. If something is deemed simple or straightforward, then it really is. If an omitted calculation is long, we say so. Details, annotations, and remarks are provided for the harder or subtler topics. Perhaps these gestures will help encourage the newly initiated to stay the course and not give up too easily.

At the end of every chapter, one finds a list of references. Other than a few books, these consist primarily of research papers mentioned in that chapter. We have chosen to list them there for a reason. It is helpful to be able to tell, at a glance, the research territories and boundaries with which the chapter in question has made contact. We hope this feature helps foster the book's image as an invitation to ongoing research. Incidentally, a master bibliography also appears at the end of the book.

We have compiled 393 exercises. Among those, there are 80 that prompt the reader to fill in some of the steps that we have omitted. Nothing was left out due to laziness on our part. Instead, the omissions are to be thought of as casualties of the editorial process. Their inclusion would either prove to be too distracting, or add unnecessarily to the size of the book. Those 80 problems aside, the remaining 313 exercises explore examples, touch upon new frontiers, and prepare for developments in later chapters.

If the purpose of the reader is to gain a nodding acquaintance of Finsler geometry, then the exercises can be skipped without harm, until some specific ones are referred to later. If the reader plans to do research in Finsler geometry, then practically all the exercises need to be carefully worked out. And, to assist those in the second group, we have provided detailed step-by-step guidance on the more challenging problems. The adventurous reader can always restore as much challenge as he or she wants by blocking out some of our suggestions. We simply want to ensure that no one feels demoralized by any of the exercises.

A good number of *explicit* examples are presented in this book. Those discussed in the sections proper include:

* Minkowski spaces: §1.3A, §14.1.
* Riemannian spaces: §13.3, especially §13.3B, §13.3C.
* Berwald spaces: §10.3, §11.6B.
* Randers spaces: §1.3C, §11.0, §11.6B, §12.6.
* Spaces of scalar curvature: §3.9B.
* Spaces of constant flag curvature: §12.6, §12.7.

Many more can be found among the exercises.

The above examples all involve *y-global* Finsler structures F, with the exception of the Berwald–Rund example treated in §10.3. By y-global, we mean that F is smooth and strongly convex on $TM \setminus 0$. The said example does not meet this stringent criterion, but is nevertheless included because it illustrates some computation well. It also provides excellent motivation for the rest of Chapter 10 and all of Chapter 11.

By no means have we exhausted the realm of interesting examples, y-global or not. For instance, it is with great reluctance that we have omitted Antonelli's Ecological Models, Matsumoto's Slope of a Mountain Metric, and Models of Physiological Optics discussed by Ingarden. The interested reader can consult the book *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology* written by these three authors.

It is true that Finsler geometry has not been nearly as popular as its progeny—Riemannian geometry. One reason is that deceptively simple formulas can quickly give rise to complicated expressions and mind-boggling computations. With the effort of many dedicated practitioners, this situation is slowly being turned around. Nonetheless, some intrinsic aspects of the subject are suggesting bounds on what one can do with mere pencil and paper.

Fortunately, we are in a technological age. Symbolic computations and large-scale computations on the computer are readily accessible. We took the first step in that direction by writing Maple codes for the Finslerian analogue of the Gaussian curvature. Then we implemented those codes on some explicit examples in Chapter 12. We hope this modest attempt represents the start of a trend. This could also be the venue by which a geometry-minded computer scientist helps advance the field significantly.

As we mentioned earlier, this book is not intended to be a comprehensive survey. Furthermore, our choice of topics and examples is guided by an eye towards the global geometry. The picture we paint can possibly be rather idiosyncratic. In spite of that, the material covered here is fundamental enough to be considered essential to all branches of Finsler geometry.

To our colleagues

In earlier versions of the manuscript, our definitions of the *nonlinear connection* and related objects on $TM \setminus 0$ differed from those of our fellow researchers by factors involving the Finsler function F. In this final version, we have decided to match their notations exactly. It is hoped that by removing an unnecessary accent, we have enhanced the book's suitability as a textbook or as a basic desk reference. Here are the specifics:

$$N^i{}_j := \gamma^i{}_{jk}\, y^k - \frac{A^i{}_{jk}}{F}\, \gamma^k{}_{rs}\, y^r\, y^s = \gamma^i{}_{jk}\, y^k - C^i{}_{jk}\, \gamma^k{}_{rs}\, y^r\, y^s ,$$

$$\frac{\delta}{\delta x^j} := \frac{\partial}{\partial x^j} - N^i{}_j\, \frac{\partial}{\partial y^i} , \qquad \delta y^i := dy^i + N^i{}_j\, dx^j .$$

We have *not* changed our philosophy of working, as much as possible, with objects that are homogeneous of degree zero in y. Our reason for doing so is that they make intrinsic sense on the manifold of rays SM. For instance, we prefer to work with $N^i{}_j/F$ rather than just $N^i{}_j$. But, unlike our earlier notation, the $N^i{}_j$ here is identical to the $N^i{}_j$ used by others.

Next, our convention on the wedge product does *not* contain the normalization factors $\frac{1}{2!}$, $\frac{1}{3!}$, etc. For example, if θ, ζ, and ξ are 1-forms, then:

$$\begin{aligned} \theta \wedge \zeta &:= \theta \otimes \zeta - \zeta \otimes \theta , \\ \theta \wedge \zeta \wedge \xi &:= \theta \otimes \zeta \otimes \xi - \theta \otimes \xi \otimes \zeta \\ &\quad + \zeta \otimes \xi \otimes \theta - \zeta \otimes \theta \otimes \xi \\ &\quad + \xi \otimes \theta \otimes \zeta - \xi \otimes \zeta \otimes \theta . \end{aligned}$$

Our placement of indices and sign convention on the curvature tensor are adequately illustrated by what we do in the Riemannian case:

$$\gamma^i{}_{jk} := \frac{g^{is}}{2} \left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right) ,$$

$$R_j{}^i{}_{kl} := \frac{\partial \gamma^i{}_{jl}}{\partial x^k} - \frac{\partial \gamma^i{}_{jk}}{\partial x^l} + \gamma^i{}_{hk}\, \gamma^h{}_{jl} - \gamma^i{}_{hl}\, \gamma^h{}_{jk} .$$

Finally, our $G^i := \gamma^i{}_{jk}\, y^j y^k$ is *twice* the G^i of Matsumoto.

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Acknowledgments

This book project began as an attempt to sort through the literature on Finsler geometry. It was our intention to write a systematic account about that part of the material which is both elementary and indispensable. We want to thank many fellow geometers for their encouragement, for answering our email calls for help, and for steering us towards the pertinent references. Some of these colleagues also helped us by proof-reading parts of the manuscript.

We are grateful to Peter Kabal (Electrical Engineering, McGill University) and Brad Lackey for the roles they played in our 20 pictorial illustrations. Brad spent countless hours researching the various software for drawing pictures. It was he who introduced us to Peter Kabal's very user-friendly TeXdraw. Brad then provided a number of examples, one of which we retained as Figure 2.1. After Brad initiated us into TeXdraw, Peter supplied us with a copy of the manual. Incidentally, Brad also alerted us to a theoretical blunder. Happily, it was something that we were able to fix.

One of us (Bao) would like to thank the University of Houston for a Limited-Grant-In-Aid (LGIA) which partially funded Brad's efforts.

There are several cherished monographs on Finsler geometry that we have not directly referenced in this book, although it is certain that we have benefited from them in many ways. We are referring to the books by Abate and Patrizio [AP], Bejancu [Bej], Miron and Anastasiei [MA].

Our book does not discuss the applications of Finsler geometry to biology, engineering, and physics. For this reason, we are especially thankful for the monographs [Asan] by Asanov, [AB] by Antonelli and Bradbury, [AIM] by Antonelli, Ingarden, and Matsumoto, and [AZas] by Antonelli and Zastawniak. We have also gained much insight from the four expository essays by Antonelli [Ant], Ingarden [Ing], Gardner and Wilkens [GW], and Beil [Bl] in the Seattle proceedings volume [BCS2].

The details for all the references cited here can be found in the master bibliography at the end of the book.

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Contents