

Robert B. Banks

Growth and Diffusion Phenomena

Mathematical Frameworks
and Applications

With 216 Figures

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Robert B. Banks

Former President and Professor
of Environmental Engineering
Asian Institute of Technology
Bangkok 10501, Thailand

Editors

F. John
Courant Institute
of Mathematical Sciences
New York University
New York, NY 10012
USA

J.E. Marsden
Department
of Mathematics
University of California
Berkeley, CA 94720
USA

L. Sirovich
Division
of Applied Mathematics
Brown University
Providence, RI 02912
USA

M. Golubitsky
Department
of Mathematics
University of Houston
Houston, TX 77004
USA

W. Jäger
Department
of Applied Mathematics
Universität Heidelberg
Im Neuenheimer Feld 294
69120 Heidelberg, FRG

Cover figure. Among the many growth and diffusion models examined by the author is one relating to the temporal-spatial growth of cities. London is utilized as an illustration. The growth of London commencing 1840 is examined in detail in Section 7.5.6

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