Graduate Texts in Mathematics **137**

Graduate Texts in Mathematics

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continued after index

Sheldon Axler Paul Bourdon
Wade Ramey

Harmonic Function Theory

With 16 Illustrations

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Sheldon Axler
Department of
Mathematics
Michigan State University
East Lansing, MI 48824
USA

Paul Bourdon
Department of
Mathematics
Washington and
Lee University
Lexington, VA 24450
USA

Wade Ramey
Department of
Mathematics
Michigan State University
East Lansing, MI 48824
USA

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Preface

Harmonic functions—the solutions of Laplace's equation—play a crucial role in many areas of mathematics, physics, and engineering. But learning about them is not always easy. At times each of the authors has agreed with Lord Kelvin and Peter Tait, who wrote ([12], Preface)

> There can be but one opinion as to the beauty and utility of this analysis of Laplace; but the manner in which it has been hitherto presented has seemed repulsive to the ablest mathematicians, and difficult to ordinary mathematical students.

The quotation has been included mostly for the sake of amusement, but it does convey a sense of the difficulties the uninitiated sometimes encounter.

The main purpose of our text, then, is to make learning about harmonic functions easier. The only prerequisite for the book is a solid foundation in real and complex analysis, together with some basic results from functional analysis. The first fifteen chapters of Rudin's *Real and Complex Analysis*, for example, provide sufficient preparation.

In several cases we simplify standard proofs. For example, we replace the usual tedious calculations showing that the Kelvin transform of a harmonic function is harmonic with some straightforward observations that we believe are more revealing. Another example is our proof of Bôcher's Theorem, which is more elementary than the classical proofs.

We also present material not usually covered in standard treatments of harmonic functions. The section on the Schwarz Lemma and the chapter on Bergman spaces are examples. For completeness, we include some topics in analysis that frequently slip through the cracks in a beginning graduate student's curriculum, such as real-analytic functions.

We rarely attempt to trace the history of the ideas presented in this book. Thus the absence of a reference does not imply originality on our part.

In addition to writing the text, the authors have developed a software package to manipulate many of the expressions that arise in harmonic function theory. Our software package, which uses many results from this book, can perform symbolic calculations that would take a prohibitive amount of time if done without a computer. For example, the Poisson integral of any polynomial can be computed exactly. Appendix B explains how readers can obtain our software package free of charge.

This book has its roots in a graduate course at Michigan State University taught by one of the authors and attended by the other authors along with a number of graduate students. The topic of harmonic functions was presented with the intention of moving on to different material after introducing the basic concepts. We did not move on to different material. Instead, we began to ask natural questions about harmonic functions. Lively and illuminating discussions ensued. A freewheeling approach to the course developed; answers to questions someone had raised in class or in the hallway were worked out and then presented in class (or in the hallway). Discovering mathematics in this way was a thoroughly enjoyable experience. We will consider this book a success if some of that enjoyment shines through in these pages.

Acknowledgments

Our book has been improved by our students. We take this opportunity to thank them for catching errors and making suggestions while attending courses at Michigan State University based on material in this book.

Among the many mathematicians who have influenced our outlook on harmonic function theory, we give special thanks to Dan Luecking for helping us to better understand Bergman spaces, and to Elias Stein and Guido Weiss for their book [10], which contributed greatly to our knowledge of spherical harmonics.

Lastly we thank the typists, who labored endlessly on this project. Although they produced some of the worst typing we have seen, the number of errors from one draft to the next did, on occasion, actually decrease. The typists were: Sheldon Axler, Paul Bourdon, and Wade Ramey.

Contents