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J. Oesterlé

A. Weinstein

Michèle Audin

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Torus Actions on
Symplectic Manifolds**

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Author's address:

Michèle Audin
IRMA Université Louis Pasteur
7 rue René Descartes
F-67084 Strasbourg Cedex
France

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to the memory of Jean Martinet

Preface

This book comes from a course I gave in Strasbourg in 1988–89. In the audience were in particular Julien Duval, Santiago Lopez de Medrano and Marcus Slupinski who helped me by their questions to understand many of the points I was supposed to be explaining. There were genuine students as well who, refusing to understand what I was badly explaining, suggested many improvements. I am thinking in particular of J. Fougeront, P. Gaucher, Li Jie and J.-M. Rinkel.

I learned a lot in discussions with Michel Brion, mainly about “Duistermaat-Heckman with singularities” and Jean-Yves MÉRINDOL, who taught me in particular the Inoue surfaces. The reader will probably think that I said almost nothing about Lie groups and algebras, she or he must know that without Nicole Bopp and Hubert Rubenthaler I would have said even less. I thank them all.

The original version of this book was published in French by the *Institut de Recherche Mathématique Avancée*. The English version the reader has in hand is not very different: I corrected some of the misprints in the mathematics, I updated the bibliographical references, I made the presence of the exercises more evident by numbering them (thanks to the automatic system of cross references in \LaTeX), I tried to suppress those of the bad jokes which were untranslatable, and I replaced a short discussion about “Duistermaat-Heckman with singularities” with a whole section in chapter V. I thank all the people who pointed out mistakes in the French version.

I thank also the Editors, who kindly accepted the book in this series and gave me good advice on the language problem, then I must thank again Marcus Slupinski who had the very difficult job of making my English readable. If there are still some gallicisms, I am the only responsible (*sic*)... and after all there are anglicisms in almost any French mathematical text!

In addition to the fact that I had to understand all the proofs and to write them down, I also typed both the French and English versions myself. I obviously cannot thank me, but I want to thank J.-M. Bony and C. Sabbah who made my work easier by their help in the use of \LaTeX .

M. A.
December 7, 1990

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