Grundlehren der mathematischen Wissenschaften 252

A Series of Comprehensive Studies in Mathematics

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Nonlinear Analysis on Manifolds. Monge–Ampère Equations

Springer–Verlag
New York Heidelberg Berlin

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AMS Subject Classification (1980): 53 CXX

Library of Congress Cataloging in Publication Data
Aubin, Thierry.
Nonlinear analysis on manifolds, Monge-Ampère equations.
(Grundlehren der mathematischen Wissenschaften; 252)
Bibliography: p.
Includes index.
1. Riemannian manifolds. 2. Monge-Ampère equations.
I. Title. II. Series.
QA649.A83 1982 516.92 82-19165

Softcover reprint of the hardcover 1st edition 1982

9 8 7 6 5 4 3 2 1

ISBN-13: 978-1-4612-5736-3 e-ISBN-13: 978-1-4612-5734-9
DOI: 10.1007/978-1-4612-5734-9

Preface

This volume is intended to allow mathematicians and physicists, especially analysts, to learn about nonlinear problems which arise in Riemannian Geometry.

Analysis on Riemannian manifolds is a field currently undergoing great development. More and more, analysis proves to be a very powerful means for solving geometrical problems. Conversely, geometry may help us to solve certain problems in analysis.

There are several reasons why the topic is difficult and interesting. It is very large and almost unexplored. On the other hand, geometric problems often lead to limiting cases of known problems in analysis, sometimes there is even more than one approach, and the already existing theoretical studies are inadequate to solve them. Each problem has its own particular difficulties.

Nevertheless there exist some standard methods which are useful and which we must know to apply them. One should not forget that our problems are motivated by geometry, and that a geometrical argument may simplify the problem under investigation. Examples of this kind are still too rare.

This work is neither a systematic study of a mathematical field nor the presentation of a lot of theoretical knowledge. On the contrary, I do my best to limit the text to the essential knowledge. I define as few concepts as possible and give only basic theorems which are useful for our topic. But I hope that the reader will find this sufficient to solve other geometrical problems by analysis.

The book is intended to be used as a reference and as an introduction to research. It can be divided into two parts, with each part containing four chapters. Part I is concerned with essential background knowledge. Part II develops methods which are applied in a concrete way to resolve specific problems.

Chapter 1 is devoted to Riemannian geometry. The specialists in analysis who do not know differential geometry will find, in the beginning of the chapter, the definitions and the results which are indispensable. Since it is useful to know how to compute both globally and in local coordinate charts, the proofs which we will present will be a good initiation. In particular, it is important to know Theorem 1.53, estimates on the components of the metric tensor in polar geodesic coordinates in terms of the curvature.

Chapter 2 studies Sobolev spaces on Riemannian manifolds. Successively, we will treat density problems, the Sobolev imbedding theorem, the Kondrakov theorem, and the study of the limiting case of the Sobolev imbedding theorem. These theorems will be used constantly. Considering the importance of Sobolev's theorem and also the interest of the proofs, three proofs of the theorem are given, the original proof of Sobolev, that of Gagliardo and Nirenberg, and my own proof, which enables us to know the value of the norm of the imbedding, an introduction to the notion of best constants in Sobolev's inequalities. This new concept is crucial for solving limiting cases.

In Chapter 3 we will find, usually without proof, a substantial amount of analysis. The reader is assumed to know this background material. It is stated here as a reference and summary of the versions of results we will be using. There are as few results as possible. I choose only the most useful and applicable ones so that the reader does not drown in a host of results and lose the main point. For instance, it is possible to write a whole book on the regularity of weak solution for elliptic equations without discussing the existence of solutions. Here there are six theorems on this topic. Of course, sometimes other will be needed; in those cases there are precise references.

It is obvious that most of the more elementary topics in this Chapter 3 have already been needed in the earlier chapters. Although we do assume prior knowledge of these basic topics, we have included precise statements of the most important concepts and facts for reference. Of course, the elementary material in this chapter could have been collected as a separate "Chapter 0" but this would have been artificial, and probably less useful to the reader. And since we do not assume that the reader knows the material on elliptic equations in Sobolev spaces, the corresponding sections should follow the two first chapters.

Chapter 4 is concerned with the Green's function of the Laplacian on compact manifolds. This will be used to obtain both some regularity results and some inequalities that are not immediate consequences of the facts in Chapter 3.

Chapter 5 provides some of the most useful methods for nonlinear analysis. As an exercise we use the variational method to solve an equation studied by Yamabe. The sketch of the proof is typical of the method. Then we solve Berger's problem and a problem posed by Nirenberg, for which we also use the results from Chapter 2 concerning the limiting case of the Sobolev imbedding theorem.

Chapter 6 is devoted to the Yamabe problem concerning the scalar curvature. Here the concept of best constants in Sobolev's inequalities plays an essential rôle. We close the chapter with a summary of the status of related problems concerning scalar curvature.

Chapter 7 is concerned with the complex Monge–Ampère equation on compact Kählerian manifolds. The existence of Einstein–Kähler metrics and the Calabi conjecture are problems which are equivalent to solving such equations.

Lastly, Chapter 8 studies the real Monge–Ampère equation on a bounded convex set of $\mathbb{R}^n$. There is also a short discussion of the complex Monge–Ampère equation on a bounded pseudoconvex set of $\mathbb{C}^n$.

Throughout the book I have restricted my attention to those problems whose solution involves typical application of the methods. Of course, there are many other very interesting problems. For example, we should at least mention that, curiously, the Yamabe equation appears in the study of Yang–Mills fields, while a corresponding complex version is very close to the existence of complex Kähler–Einstein metrics discussed in Chapter 7.

It is my pleasure and privilege to express my deep thanks to my friend Jerry Kazdan who agreed to read the manuscript from the beginning to end. He suggested many mathematical improvements, and, needless to say, corrected many blunders of mine in this English version. I also have to state in this place my appreciation for the efficient and friendly help of Jürgen Moser and Melvyn Berger for the publication of the manuscript. Pascal Cherrier and Philippe Delanoë deserve special mention for helping in the completion of the text.

May 1982 Thierry Aubin

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