Gilles Aubert  Pierre Kornprobst

# Mathematical Problems in Image Processing

## Partial Differential Equations and the Calculus of Variations

Second Edition

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Gilles Aubert
Université de Nice Sophia-Antipolis
CNRS UMR 6621
Laboratoire J.A. Dieudonné
Parc Valrose
06108 NICE CX 2
France
gaubert@math.unice.fr

Pierre Kornprobst
INRIA, Projet Odyssée
2004 route des lucioles - BP 93
06902 SOPHIA ANTIPOLIS
France
Pierre.Kornprobst@sophia.inria.fr

*Editors:*
S.S. Antman
Department of Mathematics
*and*
Institute for Physical Science and Technology
University of Maryland
College Park, MD 20742-4015
USA
ssa@math.umd.edu

J.E. Marsden
Control and Dynamical Systems, 107-81
California Institute of Technology
Pasadena, CA 91125
USA
marsden@cds.caltech.edu

L. Sirovic
Laboratory of Applied Mathematics
Department of Biomathematical Sciences
Mount Sinai School of Medicine
New York, NY 10029-6574
USA
chico@camelot.mssm.edu

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