

Søren Asmussen Peter W. Glynn

# Stochastic Simulation: Algorithms and Analysis



### ***Authors***

Søren Asmussen  
Department of Theoretical Statistics  
Department of Mathematical Sciences  
Aarhus University  
Ny Munkegade  
DK-8000 Aarhus C, Denmark  
asmus@imf.au.dk

Peter W. Glynn  
Department of Management Science  
and Engineering  
Institute for Computational and  
Mathematical Engineering  
Stanford University  
Stanford, CA 94305-4026  
glynn@stanford.edu

### ***Managing Editors***

B. Rozovskii  
Division of Applied Mathematics  
182 George St.  
Providence, RI 02912  
USA  
rozovski@dam.brown.edu

G. Grimmett  
Centre for Mathematical Sciences  
Wilberforce Road, Cambridge CB3 0WB,  
UK  
G.R.Grimmett@statslab.cam.ac.uk

Mathematics Subject Classification (2000): 65C05, 60-08, 62-01, 68-01

Library of Congress Control Number: 2007926471

ISSN: 0172-4568  
ISBN-13: 978-0-387-30679-7  
e-ISBN-13: 978-0-387-69033-9

© 2007 Springer Science+Business Media, LLC

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer Science+Business Media, LLC, 233 Spring Street, New York, NY 10013, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use in this publication of trade names, trademarks, service marks, and similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

Printed on acid-free paper.

9 8 7 6 5 4 3 2 1

[springer.com](http://springer.com)

# Contents

|  |            |
|--|------------|
| <b>Preface</b>   | <b>v</b>   |
| <b>Notation</b>  | <b>xii</b> |
| <b>I      What This Book Is About</b>                            | <b>1</b>   |
| 1     An Illustrative Example: The Single-Server Queue . . . . . | 1          |
| 2     The Monte Carlo Method . . . . .                           | 5          |
| 3     Second Example: Option Pricing . . . . .                   | 6          |
| 4     Issues Arising in the Monte Carlo Context . . . . .        | 9          |
| 5     Further Examples . . . . .                                 | 13         |
| 6     Introductory Exercises . . . . .                           | 25         |
| <b>Part A: General Methods and Algorithms</b>                    | <b>29</b>  |
| <b>II     Generating Random Objects</b>                          | <b>30</b>  |
| 1     Uniform Random Variables . . . . .                         | 30         |
| 2     Nonuniform Random Variables . . . . .                      | 36         |
| 3     Multivariate Random Variables . . . . .                    | 49         |
| 4     Simple Stochastic Processes . . . . .                      | 59         |
| 5     Further Selected Random Objects . . . . .                  | 62         |
| 6     Discrete-Event Systems and GSMPs . . . . .                 | 65         |
| <b>III    Output Analysis</b>                                    | <b>68</b>  |
| 1     Normal Confidence Intervals . . . . .                      | 68         |

|            |   |            |
|------------|---|------------|
| 2          | Two-Stage and Sequential Procedures . . . . .                       | 71         |
| 3          | Computing Smooth Functions of Expectations . . . . .                | 73         |
| 4          | Computing Roots of Equations Defined by Expectations                | 77         |
| 5          | Sectioning, Jackknifing, and Bootstrapping . . . . .                | 80         |
| 6          | Variance/Bias Trade-Off Issues . . . . .                            | 86         |
| 7          | Multivariate Output Analysis . . . . .                              | 88         |
| 8          | Small-Sample Theory . . . . .                                       | 90         |
| 9          | Simulations Driven by Empirical Distributions . . . . .             | 91         |
| 10         | The Simulation Budget . . . . .                                     | 93         |
| <b>IV</b>  | <b>Steady-State Simulation</b>                                      | <b>96</b>  |
| 1          | Introduction . . . . .  | 96         |
| 2          | Formulas for the Bias and Variance . . . . .                        | 102        |
| 3          | Variance Estimation for Stationary Processes . . . . .              | 104        |
| 4          | The Regenerative Method . . . . .                                   | 105        |
| 5          | The Method of Batch Means . . . . .                                 | 109        |
| 6          | Further Refinements . . . . .                                       | 110        |
| 7          | Duality Representations . . . . .                                   | 118        |
| 8          | Perfect Sampling . . . . .  | 120        |
| <b>V</b>   | <b>Variance-Reduction Methods</b>                                   | <b>126</b> |
| 1          | Importance Sampling . . . . .                                       | 127        |
| 2          | Control Variates . . . . .  | 138        |
| 3          | Antithetic Sampling . . . . .                                       | 144        |
| 4          | Conditional Monte Carlo . . . . .                                   | 145        |
| 5          | Splitting . . . . .   | 147        |
| 6          | Common Random Numbers . . . . .                                     | 149        |
| 7          | Stratification . . . . .  | 150        |
| 8          | Indirect Estimation . . . . .                                       | 155        |
| <b>VI</b>  | <b>Rare-Event Simulation</b>  | <b>158</b> |
| 1          | Efficiency Issues . . . . .   | 158        |
| 2          | Examples of Efficient Algorithms: Light Tails . . . . .             | 163        |
| 3          | Examples of Efficient Algorithms: Heavy Tails . . . . .             | 173        |
| 4          | Tail Estimation . . . . .   | 178        |
| 5          | Conditioned Limit Theorems . . . . .                                | 183        |
| 6          | Large-Deviations or Optimal-Path Approach . . . . .                 | 187        |
| 7          | Markov Chains and the $h$ -Transform . . . . .                      | 190        |
| 8          | Adaptive Importance Sampling via the Cross-Entropy Method . . . . . | 195        |
| 9          | Multilevel Splitting . . . . .                                      | 201        |
| <b>VII</b> | <b>Derivative Estimation</b>  | <b>206</b> |
| 1          | Finite Differences . . . . .  | 209        |
| 2          | Infinitesimal Perturbation Analysis . . . . .                       | 214        |

|  |  |            |
|--|--|------------|
| 3  | The Likelihood Ratio Method: Basic Theory . . . . .    | 220        |
| 4  | The Likelihood Ratio Method: Stochastic Processes . .  | 224        |
| 5  | Examples and Special Methods . . . . .                 | 231        |
| <b>VIII</b>                                  | <b>Stochastic Optimization</b>                         | <b>242</b> |
| 1  | Introduction . . . . .                                 | 242        |
| 2  | Stochastic Approximation Algorithms . . . . .          | 243        |
| 3  | Convergence Analysis . . . . .                         | 245        |
| 4  | Polyak–Ruppert Averaging . . . . .                     | 250        |
| 5  | Examples . . . . .                                     | 253        |
| <b>Part B: Algorithms for Special Models</b> |  | <b>259</b> |
| <b>IX</b>                                    | <b>Numerical Integration</b>                           | <b>260</b> |
| 1  | Numerical Integration in One Dimension . . . . .       | 260        |
| 2  | Numerical Integration in Higher Dimensions . . . . .   | 263        |
| 3  | Quasi-Monte Carlo Integration . . . . .                | 265        |
| <b>X</b>                                     | <b>Stochastic Differential Equations</b>               | <b>274</b> |
| 1  | Generalities about Stochastic Process Simulation . . . | 274        |
| 2  | Brownian Motion . . . . .                              | 276        |
| 3  | The Euler Scheme for SDEs . . . . .                    | 280        |
| 4  | The Milstein and Other Higher-Order Schemes . . . .    | 287        |
| 5  | Convergence Orders for SDEs: Proofs . . . . .          | 292        |
| 6  | Approximate Error Distributions for SDEs . . . . .     | 298        |
| 7  | Multidimensional SDEs . . . . .                        | 300        |
| 8  | Reflected Diffusions . . . . .                         | 301        |
| <b>XI</b>                                    | <b>Gaussian Processes</b>                              | <b>306</b> |
| 1  | Introduction . . . . .                                 | 306        |
| 2  | Cholesky Factorization. Prediction . . . . .           | 311        |
| 3  | Circulant-Embeddings . . . . .                         | 314        |
| 4  | Spectral Simulation. FFT . . . . .                     | 316        |
| 5  | Further Algorithms . . . . .                           | 320        |
| 6  | Fractional Brownian Motion . . . . .                   | 321        |
| <b>XII</b>                                   | <b>Lévy Processes</b>                                  | <b>325</b> |
| 1  | Introduction . . . . .                                 | 325        |
| 2  | First Remarks on Simulation . . . . .                  | 331        |
| 3  | Dealing with the Small Jumps . . . . .                 | 334        |
| 4  | Series Representations . . . . .                       | 338        |
| 5  | Subordination . . . . .                                | 343        |
| 6  | Variance Reduction . . . . .                           | 344        |
| 7  | The Multidimensional Case . . . . .                    | 346        |
| 8  | Lévy-Driven SDEs . . . . .                             | 348        |

|  |            |
|--|------------|
| <b>XIII Markov Chain Monte Carlo Methods</b>                         | <b>350</b> |
| 1 Introduction . . . . .   | 350        |
| 2 Application Areas . . . . .  | 352        |
| 3 The Metropolis–Hastings Algorithm . . . . .                        | 361        |
| 4 Special Samplers . . . . .   | 367        |
| 5 The Gibbs Sampler . . . . .  | 375        |
| <b>XIV Selected Topics and Extended Examples</b>                     | <b>381</b> |
| 1 Randomized Algorithms for Deterministic Optimization               | 381        |
| 2 Resampling and Particle Filtering . . . . .                        | 385        |
| 3 Counting and Measuring . . . . .                                   | 391        |
| 4 MCMC for the Ising Model and Square Ice . . . . .                  | 395        |
| 5 Exponential Change of Measure in Markov-Modulated Models . . . . . | 403        |
| 6 Further Examples of Change of Measure . . . . .                    | 407        |
| 7 Black-Box Algorithms . . . . .                                     | 416        |
| 8 Perfect Sampling of Regenerative Processes . . . . .               | 420        |
| 9 Parallel Simulation . . . . .                                      | 424        |
| 10 Branching Processes . . . . .                                     | 426        |
| 11 Importance Sampling for Portfolio VaR . . . . .                   | 432        |
| 12 Importance Sampling for Dependability Models . . . . .            | 435        |
| 13 Special Algorithms for the GI/G/1 Queue . . . . .                 | 437        |
| <b>Appendix</b>  | <b>442</b> |
| A1 Standard Distributions . . . . .                                  | 442        |
| A2 Some Central Limit Theory . . . . .                               | 444        |
| A3 FFT . . . . .   | 444        |
| A4 The EM Algorithm . . . . .  | 445        |
| A5 Filtering . . . . .   | 447        |
| A6 Itô’s Formula . . . . .   | 448        |
| A7 Inequalities . . . . .  | 450        |
| A8 Integral Formulas . . . . .                                       | 450        |
| <b>Bibliography</b>  | <b>452</b> |
| <b>Web Links</b>   | <b>469</b> |
| <b>Index</b>   | <b>471</b> |