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Random Dynamical Systems

With 40 Figures

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Preface

Background and Scope of the Book This book continues, extends, and unites various developments in the intersection of probability theory and dynamical systems. I will briefly outline the background of the book, thus placing it in a systematic and historical context and tradition.

Roughly speaking, a random dynamical system is a combination of a measure-preserving dynamical system in the sense of ergodic theory, $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}})$, $\mathbb{T} = \mathbb{R}^+$, $\mathbb{R}$, $\mathbb{Z}^+$, $\mathbb{Z}$, with a smooth (or topological) dynamical system, typically generated by a differential or difference equation $\dot{x} = f(x)$ or $x_{n+1} = \varphi(x_n)$, to a random differential equation $\dot{x} = f(\theta(t)\omega, x)$ or random difference equation $x_{n+1} = \varphi(\theta(n)\omega, x_n)$.

Both components have been very well investigated separately. However, a symbiosis of them leads to a new research program which has only partly been carried out. As we will see, it also leads to new problems which do not emerge if one only looks at ergodic theory and smooth or topological dynamics separately.

From a *dynamical systems point of view* this book just deals with those dynamical systems that have a measure-preserving dynamical system as a factor (or, the other way around, are extensions of such a factor). As there is an invariant measure on the factor, ergodic theory is always involved.

Our book is a "continuation" of that by Guckenheimer and Holmes [162] on *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* In their own words (Preface, page xi), their book "*should be seen as an attempt to extend the work of Andronov et al.* (i. e. the analysis of a single degree of freedom nonlinear oscillator, L. A.) *by one dimension* (i. e. by adding a small periodic forcing term, L. A.)". Specifically, they look at certain equations of the form $\dot{x} = f(\theta(t)\omega, x)$ in $\mathbb{R}^2$ where $\theta(t)\omega$ is periodic. We will go further and beyond the periodic "noise" to a general measure-preserving dynamical system to which the ordinary differential equation is coupled. In yet other words, we take the step from autonomous systems $\dot{x} = f(x)$ to nonautonomous systems, but of the special kind $\dot{x} = f(\theta(t)\omega, x)$, i. e. to those which are coupled to a dynamical "bath".

If the flow $\omega \mapsto \theta(t)\omega$ in the equation $\dot{x} = f(\theta(t)\omega, x)$ is a flow of homeomorphisms of a compact space we are in the realm of skew-product flows in the sense of Sacker, Sell and Johnson (see e.g. [187], [298], [299], [316]). We go beyond this again by stripping off all the topology from $\omega \mapsto \theta(t)\omega$, and instead adding an invariant measure – shortly, by going from "almost periodic" to "random".

We also extend and generalize Mañé's book [249] on *Ergodic Theory and Differentiable Dynamics.* He has a measure ρ invariant with respect to the flow $\varphi(t)$ of a deterministic vector field $\dot{x} = f(x)$ on a manifold M. Here, we have a measure μ on $\Omega \times M$ with marginal $\mathbb{P}$ on Ω invariant with respect to the skew-product flow $(\omega, x) \mapsto (\theta(t)\omega, \varphi(t,\omega)x)$, where $\varphi(t,\omega)$ is the solution flow generated by the random vector field $\dot{x} = f(\theta(t)\omega, x)$.

From a *probabilistic point of view* this book offers another look at the quite classical subject of random difference equations and of random and stochastic differential equations, i. e. ordinary differential equations driven by real or white noise.

During the last 20 to 30 years an impressive structure called "stochastic analysis" has been erected, part of which is a theory of differential equations with semimartingale (rather than only Gaussian white noise, or Wiener) driving processes, providing us with a unified theory of random and stochastic differential equations.

Around 1980 it was discovered by Elworthy, Baxendale, Bismut, Ikeda, Watanabe, Kunita and others (see e.g. [137], [53], [72], [178], [223]) that a stochastic differential equation generates "for free" a much richer structure than just a family of stochastic processes, each solving the stochastic differential equation for a given initial value. It gives us in fact a flow of random diffeomorphisms. We can now bridge the gap between stochastic analysis and dynamical systems by proving that a random or stochastic differential equation generates a random dynamical system.

This makes it possible to re-evaluate and improve all the classical results (which are based on one-point motions and Markov transition probabilities) on stochastic stability, existence of invariant measures, etc. by Kushner [225], Khasminskii [206], Bunke [84] and many others. In [8] I have described the extension of the horizon when going from Markov processes to stochastic flows and cocycles.

The present book also adds a new chapter to the volume by Horsthemke and Lefever [175] entitled *Noise-Induced Transitions* and re-interprets their findings: Their noise-induced transitions are nothing but bifurcations on the static level of the Fokker-Planck equation. We will also study bifurcation scenarios on the dynamic level.

The book closest to ours in spirit and content is the one by Kifer [207] on *Ergodic Theory of Random Transformations.* He, however, deals exclusively with the i.i.d. case, i. e. with the case of iterations of random mappings chosen independently with identical distribution. In this case the orbits in state space form a Markov chain. We go beyond that by allowing a stationary stochastic sequence of mappings to be iterated, keeping the i.i.d. case as an important particular case.

It is a characteristic feature of the theory of random dynamical systems that every problem involves some ergodic theory and ergodic theorems. The most crucial and most important ergodic theorem applies to the linearization of smooth random dynamical systems. It is traditionally called the *Multiplicative Ergodic Theorem* and was proved by Oseledets [268] in 1968. This theorem provides a random substitute of linear algebra and hence makes a local theory of smooth random dynamical systems possible. Without it the whole field (in particular this book) would not exist.

Structure of the Book As this is the first monograph on random dynamical systems, my main intention is foundational. This forces me to adopt a systematic, maybe sometimes even pedantic, style, and put my emphasis on theory rather than applications. I hope nevertheless to present a useful, reliable, and rather complete source of reference which lays the foundations for future work and applications.

Part I (Random Dynamical Systems and Their Generators) introduces the subject matter, settles the subtle perfection question, develops the theory of invariant measures (Chap. 1) and gives a (hopefully ultimate) treatment of the problem of which random dynamical systems have infinitesimal generators (Chap. 2).

Part II (Multiplicative Ergodic Theory) is the heart of the book. I first present and prove the classical Multiplicative Ergodic Theorem for products of random matrices in $\mathbb{R}^d$ (Chap. 3), then present its various modifications and the concept of random norms which turns out to be basic (Chap. 4). In Chap. 5 the multiplicative ergodic theory of related linear random dynamical systems obtained by taking the inverse, the adjoint, and exterior and tensor products is studied. The same is done with the systems induced by a linear random dynamical system on the unit sphere, the projective space, and Grassmannian manifolds, culminating in the Furstenberg-Khasminskii formulas for Lyapunov exponents. Finally, a multiplicative ergodic theorem for rotation numbers is proved (Chap. 6).

Part III (Smooth Random Dynamical Systems) addresses the three most fundamental problems regarding nonlinear systems. The first one is the construction of invariant manifolds (Chap. 7). We adopt the new method of Wanner [340] which provides a unified approach towards invariant manifolds and the Hartman-Grobman theorem. The second basic problem is the simplification of a random dynamical system by means of a smooth coordinate transformation (normal form problem) (Chap. 8). I finally present the state of the art of random bifurcation theory (Chap. 9) which is still in its infancy and is not much more than a collection of (numerical) examples.

Part IV (Appendices) collects some facts from measurable dynamics (Appendix A) and smooth dynamics (Appendix B).

This book is a research monograph which belongs to the richly structured interface of probability theory and dynamical systems. Therefore, substantial mathematical knowledge is required from the reader.

The book as a whole is probably not suitable for a course. There are, however, various possibilities to use subsets of the text as the basis of a graduate course or seminar or for private study. For example:

(i) The multiplicative ergodic theorem: Extract the basic definitions from Chap. 1, then read Chaps. 3 and 4.

(ii) Smooth random dynamical systems: Read Chaps. 1, 3 and 4, then choose one or several of the Chaps. 7, 8, or 9.

(iii) Stochastic bifurcation theory: Read Chaps. 1, 3 and 4, then go to Chap. 9.

Omissions The exclusion of the following topics from this book is partly compensated by some recent publications which augment this book and complete the overall picture of the subject.

I completely omitted topological dynamics of random dynamical systems and refer instead to the book by Nguyen Dinh Cong [261].

Fortunately, I do not need to include *Pesin's theory*, probably the deepest of recent developments in random dynamical systems, as it is beautifully and completely presented in the book by Liu and Qian [244].

With many scruples, I decided to omit the beautiful "geometry of stochastic flows" (see the work of Baxendale [56, 55, 59, 61, 62], Baxendale and Stroock [64], Carverhill [91], Carverhill and Elworthy [95], Elworthy [139, 140, 141], Elworthy and Rosenberg [144], Elworthy and Yor [145], Elworthy, Le Jan and Li [142], Elworthy and Li [143], Kunita [224: section 4.9] Li [234], Liao [236, 237, 238, 239], and the references therein). The subject is worth a book of its own.

I also did not include the theory proper of products of random matrices. On the one hand, this area with its elaborate methods and numerous applications is so vast that it would easily fill a volume in itself. On the other hand, the subject is already quite well-documented: Besides the books by Bougerol and Lacroix [77] and Högnäs and Mukherjea [172], there are numerous contributions, survey articles, and further references in the three proceedings volumes [104], [39], and [14].

Finally, I omitted infinite-dimensional random systems and instead refer the reader to the work of Crauel and Flandoli [117, 119], Flandoli and Schmalfuß [151], Mohammed [255, 256], Mohammed and Scheutzow [257], Schaumlöffel [301], in addition to many others.

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Bremen, May 1998 *Ludwig Arnold*

Contents

Part I. Random Dynamical Systems and Their Generators

Part IV. Appendices