

# Undergraduate Texts in Mathematics

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## Undergraduate Texts in Mathematics

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- Abbott:** Understanding Analysis.
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- Hairer/Wanner:** Analysis by Its History.  
*Readings in Mathematics.*

(continued after index)

M. A. Armstrong

# Basic Topology

With 132 Illustrations



Springer

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*Dedicated to the memory of*

PAUL DUFÉTELLE

# Preface to the Springer Edition

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This printing is unchanged, though the opportunity has been taken to correct one or two misdemeanours. In particular Problems 2.13, 3.13 and 3.19 are now correctly stated, and Tietze has regained his final “e”. My thanks go to Professor P. R. Halmos and to Springer-Verlag for the privilege of appearing in this series.

M.A.A.  
Durham, January 1983.

# Preface

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This is a topology book for undergraduates, and in writing it I have had two aims in mind. Firstly, to make sure the student sees a variety of different techniques and applications involving point set, geometric, and algebraic topology, without delving too deeply into any particular area. Secondly, to develop the reader's geometrical insight; topology is after all a branch of geometry.

The prerequisites for reading the book are few, a sound first course in real analysis (as usual!), together with a knowledge of elementary group theory and linear algebra. A reasonable degree of 'mathematical maturity' is much more important than any previous knowledge of topology.

The layout is as follows. There are ten chapters, the first of which is a short essay intended as motivation. Each of the other chapters is devoted to a single important topic, so that identification spaces, the fundamental group, the idea of a triangulation, surfaces, simplicial homology, knots and covering spaces, all have a chapter to themselves.

Some motivation is surely necessary. A topology book at this level which begins with a set of axioms for a topological space, as if these were an integral part of nature, is in my opinion doomed to failure. On the other hand, topology should not be presented as a collection of party tricks (colouring knots and maps, joining houses to public utilities, or watching a fly escape from a Klein bottle). These things all have their place, but they must be shown to fit into a unified mathematical theory, and not remain dead ends in themselves. For this reason, knots appear at the end of the book, and not at the beginning. It is not the knots which are so interesting, but rather the variety of techniques needed to deal with them.

Chapter 1 begins with Euler's theorem for polyhedra, and the theme of the book is the search for topological invariants of spaces, together with techniques for calculating them. Topology is complicated by the fact that something which is, by its very nature, topologically invariant is usually hard to calculate, and vice versa the invariance of a simple number like the Euler characteristic can involve a great deal of work.

The balance of material was influenced by the maxim that a theory and its payoff in terms of applications should, wherever possible, be given equal weight. For example, since homology theory is a good deal of trouble to set up (a whole chapter), it must be shown to be worth the effort (a whole chapter of applications). Moving away from a topic is always difficult, and the temptation to include more and more is hard to resist. But to produce a book of reasonable length some topics just have to go; I mention particularly in this respect the omission of any systematic method for calculating homology groups. In

## PREFACE

formulating definitions, and choosing proofs, I have not always taken the shortest path. Very often the version of a definition or result which is most convenient to work with, is not at all natural at first sight, and this is above all else a book for beginners.

Most of the material can be covered in a one-year course at third-year (English) undergraduate level. But there is plenty of scope for shorter courses involving a selection of topics, and much of the first half of the book can be taught to second-year students. Problems are included at the end of just about every section, and a short bibliography is provided with suggestions for parallel reading and as to where to go next.

The material presented here is all basic and has for the most part appeared elsewhere. If I have made any contribution it is one of selection and presentation.

Two topics deserve special mention. I first learned about the Alexander polynomial from J. F. P. Hudson, and it was E. C. Zeeman who showed me how to do surgery on surfaces. To both of them, and particularly to Christopher Zeeman for his patience in teaching me topology, I offer my best thanks.

I would also like to thank R. S. Roberts and L. M. Woodward for many useful conversations, Mrs J. Gibson for her speed and skill in producing the manuscript, and Cambridge University Press for permission to reproduce the quotation from Hardy's 'A Mathematician's Apology' which appears at the beginning of Chapter 1. Finally, a special word of thanks to my wife Anne Marie for her constant encouragement.

M.A.A.

Durham, July 1978.



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