

Nonstandard Analysis

Theory and Applications

edited by

Leif O. Arkeryd

University of Gothenburg, Sweden

Nigel J. Cutland

University of Hull, England

and

C. Ward Henson

University of Illinois,
Urbana-Champaign, U.S.A.



Springer-Science+Business Media, B.V.

Proceedings of the NATO Advanced Study Institute on
Nonstandard Analysis and its Applications
International Centre for Mathematical Sciences, Edinburgh, Scotland
30 June – 13 July 1996

A C.I.P. Catalogue record for this book is available from the Library of Congress

ISBN 978-94-010-6335-7 ISBN 978-94-011-5544-1 (eBook)
DOI 10.1007/978-94-011-5544-1

Printed on acid-free paper

All Rights Reserved

© 1997 Springer Science+Business Media Dordrecht

Originally published by Kluwer Academic Publishers in 1997

Softcover reprint of the hardcover 1st edition 1997

No part of the material protected by this copyright notice may be reproduced or utilized in any form or by any means, electronic or mechanical, including photocopying, recording or by any information storage and retrieval system, without written permission from the copyright owner.

CONTENTS

PREFACE	ix
FOUNDATIONS OF NONSTANDARD ANALYSIS	1
<i>A Gentle Introduction to Nonstandard Extensions</i>	
C. WARD HENSON	
1 Introduction	1
2 Nonstandard Extensions	2
3 Logical Formulas	14
4 Nonstandard Extensions of Multisets	22
5 Nonstandard Extensions of the Multiset $(X, \mathcal{P}(X))$	29
6 Superstructures	36
7 Saturation	43
NONSTANDARD REAL ANALYSIS	51
NIGEL J. CUTLAND	
1 Introduction	51
2 Basic Properties of ${}^*\mathbb{R}$	51
3 Sequences and Series	56
4 Continuity	59
5 Differentiation	62
6 Riemann Integration	64
7 Topology on \mathbb{R}	67
8 Using Internal Subsets of ${}^*\mathbb{R}$	69
9 An Application: Differential Equations	75

NONSTANDARD ANALYSIS AND TOPOLOGY 77

PETER A. LOEB

1	Metric and Topological Spaces	77
2	Continuous Mappings	79
3	Convergence	80
4	More on Topologies	81
5	Compact Spaces	82
6	Product Spaces	84
7	Restricted or Relative Topologies	84
8	Uniform Continuity on Metric Spaces	84
9	Nonstandard Hulls	85
10	Compactifications	86
11	More Exercises	86

LOEB MEASURE AND PROBABILITY 91

DAVID A. ROSS

1	Introduction	91
2	Finite Loeb Measure	94
3	Constructing Standard Measures	98
4	Representing Standard Measures	102
5	Measurable Functions	105
6	Integration Theory	109
7	Probability Theory	112
8	Advertisement	115
9	Exercises	116

**AN INTRODUCTION TO NONSTANDARD
FUNCTIONAL ANALYSIS** 121

MANFRED P. H. WOLFF

1	Elementary Nonstandard Analysis of Normed Linear Spaces	121
2	Advanced Theory of Banach Spaces	130
3	Elementary Theory of Linear Operators	135
4	Spectral Theory of Bounded Operators	140
5	Applications of Nonstandard Spectral Theory	143
6	Closed Operators	146

**APPLICATIONS OF NONSTANDARD ANALYSIS
IN ORDINARY DIFFERENTIAL EQUATIONS** **153**

E. BENOIT

1	Introduction	153
2	Tools in NSA	153
3	Differential Equations and Recursive Sequences	159
4	Regular Perturbations	167
5	Example	169
6	Dynamical Systems: Notions of Stability	172
7	Singular Perturbations	174

**BETTER NONSTANDARD UNIVERSES
WITH APPLICATIONS** **183**

R. JIN

1	Introduction	183
2	The Isomorphism Property	186
3	The Special Model Axiom and Full Saturation	199
4	The λ -Bolzano-Weierstrass Property	205

**INTERNAL MARTINGALES AND
STOCHASTIC INTEGRATION** **209**

TOM LINDSTRØM

1	Hyperfinite Probability Spaces	210
2	Poisson Processes	213
3	Brownian Motion	216
4	Internal Martingales	220
5	Doob's Inequality	223
6	Quadratic Variation	225
7	Standard Parts	228
8	S-continuity	231
9	Stochastic Integration	237
10	Itô's Formula	240
11	Lévy's Characterization of Brownian Motion	242
12	Connections to Standard Theory	244
13	Stochastic Integrals in Higher Dimensions	249
14	Stochastic Differential Equations	251
15	Brownian Local Time	253
16	The Infinite Dimensional Ornstein-Uhlenbeck Process	254

STOCHASTIC DIFFERENTIAL EQUATIONS WITH EXTRA PROPERTIES	259
H. JEROME KEISLER	
1 Introduction	259
2 Spaces of Stochastic Processes	260
3 Solutions of Stochastic Differential Equations	263
4 Solutions which are Markov Processes	270
5 A Fixed Point Theorem	273
6 Stochastic Differential Equations with Nondegenerate Coefficients	276
HYPERFINITE MATHEMATICAL FINANCE	279
P. EKKEHARD KOPP	
1 Introduction	279
2 Finite Market Models	283
3 Pricing Options in a Hyperfinite CRR Model	289
4 Hyperfinite Trading Strategies	293
5 Convergence of Prices and Strategies	296
6 Further Developments	304
APPLICATIONS OF NSA TO MATHEMATICAL PHYSICS	309
LEIF ARKERYD	
1 A Kinetic Inequality	310
2 The Time Asymptotic Behaviour for Certain Rarefied Gases when the Incoming Fluxes at the Boundary are Given	317
3 On Semiclassical Limits for the Schrödinger Equation	326
A NONSTANDARD APPROACH TO HYDROMECHANICS	341
<i>Navier–Stokes Equations</i>	
M. CAPIŃSKI	
1 Introduction	341
2 Deterministic Navier–Stokes Equations	344
3 Statistical Solutions	350
4 Stochastic Equations	354
5 Some Open Problems	355
INDEX	357