

Peter J. Olver

Applications of Lie Groups to Differential Equations

Second Edition



Springer

Peter J. Olver
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
USA

Editorial Board

S. Axler
Department of
Mathematics
San Francisco State
University
San Francisco, CA 94132
USA

F.W. Gehring
Department of
Mathematics
University of Michigan
Ann Arbor, MI 48109
USA

K.A. Ribet
Department of
Mathematics
University of California
at Berkeley
Berkeley, CA 94720
USA

With 10 illustrations.

Mathematics Subject Classifications (1991): 22E70, 34-01, 70H05

Library of Congress Cataloging-in-Publication Data

Olver, Peter J.

Applications of Lie groups to differential equations / Peter J.

Olver.—2nd ed.

p. cm.—(Graduate texts in mathematics; 107)

Includes bibliographical references and indexes.

ISBN-13: 978-0-387-95000-6

e-ISBN-13: 978-1-4612-4350-2

DOI: 10.1007/978-1-4612-4350-2

1. Differential equations. 2. Lie groups. I. Title.

II. Series.

QA372.055 1993

515'.35—dc20

92-44573

Printed on acid-free paper.

First softcover printing, 2000.

© 1986, 1993 Springer-Verlag New York, Inc.

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

Production managed by Jim Harbison, manufacturing supervised by Vincent Scelta.

Typeset by Asco Trade Typesetting Ltd., Hong Kong.

9 8 7 6 5 4 3 2 1

SPIN 10755712

Table of Contents

Preface to First Edition	v
Preface to Second Edition	vii
Acknowledgments	ix
Introduction	xvii
Notes to the Reader	xxv

CHAPTER 1

Introduction to Lie Groups	1
1.1. Manifolds	2
Change of Coordinates	6
Maps Between Manifolds	7
The Maximal Rank Condition	7
Submanifolds	8
Regular Submanifolds	11
Implicit Submanifolds	11
Curves and Connectedness	12
1.2. Lie Groups	13
Lie Subgroups	17
Local Lie Groups	18
Local Transformation Groups	20
Orbits	22
1.3. Vector Fields	24
Flows	27
Action on Functions	30
Differentials	32
Lie Brackets	33
Tangent Spaces and Vectors Fields on Submanifolds	37
Frobenius' Theorem	38

1.4. Lie Algebras	42
One-Parameter Subgroups	44
Subalgebras	46
The Exponential Map	48
Lie Algebras of Local Lie Groups	48
Structure Constants	50
Commutator Tables	50
Infinitesimal Group Actions	51
1.5. Differential Forms	53
Pull-Back and Change of Coordinates	56
Interior Products	56
The Differential	57
The de Rham Complex	58
Lie Derivatives	60
Homotopy Operators	63
Integration and Stokes' Theorem	65
Notes	67
Exercises	69
 CHAPTER 2	
Symmetry Groups of Differential Equations	75
2.1. Symmetries of Algebraic Equations	76
Invariant Subsets	76
Invariant Functions	77
Infinitesimal Invariance	79
Local Invariance	83
Invariants and Functional Dependence	84
Methods for Constructing Invariants	87
2.2. Groups and Differential Equations	90
2.3. Prolongation	94
Systems of Differential Equations	96
Prolongation of Group Actions	98
Invariance of Differential Equations	100
Prolongation of Vector Fields	101
Infinitesimal Invariance	103
The Prolongation Formula	105
Total Derivatives	108
The General Prolongation Formula	110
Properties of Prolonged Vector Fields	115
Characteristics of Symmetries	115
2.4. Calculation of Symmetry Groups	116
2.5. Integration of Ordinary Differential Equations	130
First Order Equations	131
Higher Order Equations	137
Differential Invariants	139
Multi-parameter Symmetry Groups	145
Solvable Groups	151
Systems of Ordinary Differential Equations	154

2.6. Nondegeneracy Conditions for Differential Equations	157
Local Solvability	157
Invariance Criteria	161
The Cauchy–Kovalevskaya Theorem	162
Characteristics	163
Normal Systems	166
Prolongation of Differential Equations	166
Notes	172
Exercises	176
 CHAPTER 3	
Group-Invariant Solutions	183
3.1. Construction of Group-Invariant Solutions	185
3.2. Examples of Group-Invariant Solutions	190
3.3. Classification of Group-Invariant Solutions	199
The Adjoint Representation	199
Classification of Subgroups and Subalgebras	203
Classification of Group-Invariant Solutions	207
3.4. Quotient Manifolds	209
Dimensional Analysis	214
3.5. Group-Invariant Prolongations and Reduction	217
Extended Jet Bundles	218
Differential Equations	222
Group Actions	223
The Invariant Jet Space	224
Connection with the Quotient Manifold	225
The Reduced Equation	227
Local Coordinates	228
Notes	235
Exercises	238
 CHAPTER 4	
Symmetry Groups and Conservation Laws	242
4.1. The Calculus of Variations	243
The Variational Derivative	244
Null Lagrangians and Divergences	247
Invariance of the Euler Operator	249
4.2. Variational Symmetries	252
Infinitesimal Criterion of Invariance	253
Symmetries of the Euler–Lagrange Equations	255
Reduction of Order	257
4.3. Conservation Laws	261
Trivial Conservation Laws	264
Characteristics of Conservation Laws	266
4.4. Noether’s Theorem	272
Divergence Symmetries	278
Notes	281
Exercises	283

CHAPTER 5

Generalized Symmetries	286
5.1. Generalized Symmetries of Differential Equations	288
Differential Functions	288
Generalized Vector Fields	289
Evolutionary Vector Fields	291
Equivalence and Trivial Symmetries	292
Computation of Generalized Symmetries	293
Group Transformations	297
Symmetries and Prolongations	300
The Lie Bracket	301
Evolution Equations	303
5.2. Recursion Operators, Master Symmetries and Formal Symmetries	304
Fréchet Derivatives	307
Lie Derivatives of Differential Operators	308
Criteria for Recursion Operators	310
The Korteweg–de Vries Equation	312
Master Symmetries	315
Pseudo-differential Operators	318
Formal Symmetries	322
5.3. Generalized Symmetries and Conservation Laws	328
Adjoints of Differential Operators	328
Characteristics of Conservation Laws	330
Variational Symmetries	331
Group Transformations	333
Noether's Theorem	334
Self-adjoint Linear Systems	336
Action of Symmetries on Conservation Laws	341
Abnormal Systems and Noether's Second Theorem	342
Formal Symmetries and Conservation Laws	346
5.4. The Variational Complex	350
The D-Complex	351
Vertical Forms	353
Total Derivatives of Vertical Forms	355
Functionals and Functional Forms	356
The Variational Differential	361
Higher Euler Operators	365
The Total Homotopy Operator	368
Notes	374
Exercises	379

CHAPTER 6

Finite-Dimensional Hamiltonian Systems	389
6.1. Poisson Brackets	390
Hamiltonian Vector Fields	392
The Structure Functions	393
The Lie–Poisson Structure	396

6.2. Symplectic Structures and Foliations	398
The Correspondence Between One-Forms and Vector Fields	398
Rank of a Poisson Structure	399
Symplectic Manifolds	400
Maps Between Poisson Manifolds	401
Poisson Submanifolds	402
Darboux' Theorem	404
The Co-adjoint Representation	406
6.3. Symmetries, First Integrals and Reduction of Order	408
First Integrals	408
Hamiltonian Symmetry Groups	409
Reduction of Order in Hamiltonian Systems	412
Reduction Using Multi-parameter Groups	416
Hamiltonian Transformation Groups	418
The Momentum Map	420
Notes	427
Exercises	428
 CHAPTER 7	
Hamiltonian Methods for Evolution Equations	433
7.1. Poisson Brackets	434
The Jacobi Identity	436
Functional Multi-vectors	439
7.2. Symmetries and Conservation Laws	446
Distinguished Functionals	446
Lie Brackets	446
Conservation Laws	447
7.3. Bi-Hamiltonian Systems	452
Recursion Operators	458
Notes	461
Exercises	463
 References	467
Symbol Index	489
Author Index	497
Subject Index	501