Progress in Mathematics
Volume 103

Series Editors
J. Oesterlé
A. Weinstein

Bernd Anger
Claude Portenier

Radon Integrals

An abstract approach to integration and Riesz representation through function cones

Springer Science+Business Media, LLC

Bernd Anger
Mathematisches Institut
Universität Erlangen-Nürnberg
8520 Erlangen
Germany

Claude Portenier
Fachbereich Mathematik
Universität Marburg
3550 Marburg
Germany

Printed on acid-free paper.

© Springer Science+Business Media New York 1992.
Originally published by Birkhäuser Boston in 1992
Softcover reprint of the hardcover 1st edition 1992
All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without prior permission of the copyright owner.
Permission to photocopy for internal or personal use of specific clients is granted by Springer Science+Business Media, LLC,
for libraries and other users registered with the Copyright Clearance
Center (CCC), provided that the base fee of $0.00 per copy, plus $0.20 per page is paid directly to CCC, 21 Congress Street, Salem, MA 01970, U.S.A. Special requests should be addressed directly to Springer Science+Business Media, LLC.

ISBN 978-1-4612-6733-1 ISBN 978-1-4612-0377-3 (eBook)
DOI 10.1007/978-1-4612-0377-3

Camera-ready text prepared by the Authors.

9 8 7 6 5 4 3 2 1

PREFACE

In topological measure theory, Radon measures are the most important objects. In the context of locally compact spaces, there are two equivalent canonical definitions. As a set function, a Radon measure is an inner compact regular Borel measure, finite on compact sets. As a functional, it is simply a positive linear form, defined on the vector lattice of continuous real-valued functions with compact support.

During the last few decades, in particular because of the developments of modern probability theory and mathematical physics, attention has been focussed on measures on general topological spaces which are no longer locally compact, e.g. spaces of continuous functions or Schwartz distributions.

For a *Radon measure on an arbitrary Hausdorff space*, essentially three equivalent definitions have been proposed :

As a set function, it was defined by L. Schwartz as an inner compact regular Borel measure which is locally bounded. G. Choquet considered it as a strongly additive right continuous content on the lattice of compact subsets. Following P.A. Meyer, N. Bourbaki defined a Radon measure as a locally uniformly bounded family of compatible positive linear forms, each defined on the vector lattice of continuous functions on some compact subset.

Compared with the simplicity of the functional analytic description in the locally compact case, it seems that the "linear functional aspect" of Radon measures has been lost in the general situation. It is our aim to show how to define "Radon integrals" as certain linear functionals, and then how to develop a theory of integration in a functional analytic spirit. Obviously, the vector lattice of continuous functions with compact support is too small, possibly even degenerate, to serve as the domain of a Radon integral. As a substitute, we consider the function cone $\mathcal{I}(X)$, i.e. the positively homogeneous and additive set of all lower semicontinuous functions on a Hausdorff space X which take values in $\widetilde{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ and are positive outside a suitable compact set. The integral on $\mathcal{I}(X)$ with respect to a Radon measure is an increasing linear $\widetilde{\mathbb{R}}$-valued functional, i.e. it is positively homogeneous and additive. Among all these functionals, Radon integrals are char–

acterized by the following regularity property, which reflects the inner regularity of Radon measures :

For every function $s \in \mathcal{S}(X)$, the integral of s can be approximated from below by the integrals of minorants of s in $-\mathcal{S}(X)$.

This leads us to define *Radon integrals* on X as regular linear functionals on $\mathcal{S}(X)$. The concept of integrability is introduced by the coincidence and finiteness of the upper and lower integrals, defined by approximation from above with functions in $\mathcal{S}(X)$, respectively from below with functions in $-\mathcal{S}(X)$. Apart from the asymmetry of approximation, this is a kind of *abstract Riemann* , i.e. finitely additive, integration theory.

One of the advantages of our simple description of Radon integrals is its extendability to functionals on cones of semicontinuous sections in a non-trivial line bundle, which may be used to treat the concept of conical measures. Note that in this context, there is no adequate set-theoretical notion of measure. However, this generalization would go beyond the limits of our exposition.

On the other hand, since a large part of the theory of integration depends only on the regularity property, the consideration of regular linear functionals on *arbitrary function cones* needs no additional tools, but enriches the theory considerably. It allows us to treat Radon integrals as a fundamental example and applies simultaneously to Radon measures in the sense of Choquet, to abstract set-theoretical aspects of integration with respect to contents on lattices of sets, to Loomis' abstract Riemann integration for positive linear forms on vector lattices of real-valued functions, as well as to the Daniell and Bourbaki integration theories. Furthermore, we have in mind future applications to potential theory and convexity, in particular Choquet representation theory. Typically, the function cones to be considered there are min-stable but not lattice cones.

We present a unified functional analytic approach to integration, in an abstract Riemann spirit, based on the following two fundamental objects :

a min-stable *cone* $\mathcal{S}$ of $\widetilde{\mathbb{R}}$-valued functions on an arbitrary set X ,
a *regular* linear $\widetilde{\mathbb{R}}$-valued functional on $\mathcal{S}$.

Note that a positive linear form μ on a vector lattice of functions is always regular, whereas on a cone of positive functions the only regular linear functional is the trivial one. It is therefore just the asymmetry of function *cones* that enables us to treat regularity. On the other hand, for Daniell and Bourbaki integration, even if $\mathcal{S}$ is a vector lattice, one has to consider the function cones $\mathcal{S}_\sigma$ and $\mathcal{S}_\phi$ of all upper envelopes of sequences, respectively families of functions in $\mathcal{S}$. Using the relevant convergence properties, one first extends μ to a regular linear functional on these cones and then applies our abstract Riemann theory to the extensions.

Even for *Riesz representation theorems*, concentration on the abstract Riemann theory achieves clarity, convergence properties playing no role at all. In our abstract setting, we study the representability of a linear functional τ on a function cone $\mathcal{T}$ as an abstract Riemann integral with respect to another linear functional μ on a given function cone $\mathcal{S}$. Since the functions in $\mathcal{T}$ may be $\mathcal{S}$–unbounded, whereas integrable functions with respect to μ are always $\mathcal{S}$–bounded, we have to develop a suitable theory of *essential integration*.

The cornerstone of our approach to integration and essential integration is the concept of an *upper functional*. This is an abstract version of an upper integral, possibly without convergence properties. The general theory of integration based on this concept is developed in the first chapter.

For Riesz representation theorems, which together with regularity and Radon integrals are the main topics of the second chapter, there are two more indispensable fundamental notions to be mentioned. The first one, *tightness*, controls the representability as well as the manner of representation. The second notion, a certain kind of *measurability* with respect to the cone $\mathcal{S}$, is needed since the functions in $\mathcal{T}$ have to be measurable for the initially unknown representing functional.

As a fundamental result, we prove that every tight regular linear functional defined on a sufficiently rich lattice cone of lower semicontinuous functions on a Hausdorff space is represented by a unique Radon integral. This is a fairly general representation theorem with Radon integrals.

As mentioned before, we also treat *set–theoretical aspects* of integration for contents, defined on lattices of sets. We incorporate this abstract measure theory by first proving a correspondence between regular contents and regular linear functionals defined on a suitable cone of step functions. In particular, we infer that the set–theoretical counterparts of Radon integrals are *Radon measures*, i.e. finite contents on the lattice of compact subsets, regular with respect to the lattice of open subsets. Secondly, we introduce an adequate set–theoretical concept of *measurability* , more general than that of Loomis. We show its equivalence to the functional analytic concept of M.H. Stone which is defined by the property that truncation by integrable functions leads to integrable functions.

In order to get representation by contents, we show more generally that measurability with respect to the cone of step functions is equivalent to lattice–measurability, a concept which for a δ–lattice coincides with semicontinuity.

It was not our aim to treat all aspects of integration theory. But we hope that our functional analytic approach to integration, and in particular to Radon integrals, will lead to fruitful further developments.

We would like to express our deep gratitude to Prof. R. B. Burckel for his kind and helpful assistance in revising the English version of the manuscript. We are very much indepted to Prof. H. Bauer and Prof. J. Lembcke for encouraging conversations and stimulating discussions on the subject, as well as to our students for suggesting valuable improvements. We also thank Frau J. Happel for typing previous versions of the manuscript with great patience.

Erlangen and Marburg
November 1991

Bernd Anger
Claude Portenier

CONTENTS