Ergebnisse der Mathematik und ihrer Grenzgebiete

Volume 33

3. Folge

A Series of Modern Surveys in Mathematics

Editorial Board

E. Bombieri, Princeton S. Feferman, Stanford
M. Gromov, Bures-sur-Yvette J. Jost, Bochum
H.W. Lenstra, Jr., Berkeley P.-L. Lions, Paris
R. Remmert (Managing Editor), Münster
W. Schmid, Cambridge, Mass. J-P. Serre, Paris
J.Tits, Paris

Springer
Berlin
Heidelberg
New York
Barcelona
Budapest
Hong Kong
London
Milan
Paris
Santa Clara
Singapore
Tokyo

Carlos Andradas
Ludwig Bröcker
Jesús M. Ruiz

Constructible Sets in Real Geometry

Springer

Carlos Andradas
Departamento de Algebra
Universidad Complutense de Madrid
E-28040 Madrid
Spain
e-mail: carlos@sunal1.mat.ucm.es

Ludwig Bröcker
Mathematisches Institut
Universität Münster
Einsteinstraße 62
D-48149 Münster
Germany
e-mail: broe@escher.uni-muenster.de

Jesús M. Ruiz
Departamento de Geometría y Topología
Universidad Complutense de Madrid
E-28040 Madrid
Spain
e-mail: jesusr@eucmax.sim.ucm.es

Library of Congress Cataloging-in-Publication Data
Andradas, Carlos.
Constructible sets in real geometry / Carlos Andradas, Ludwig Bröcker, Jesús M. Ruiz.
p. cm. -- (Ergebnisse der Mathematik und ihrer Grenzgebiete ; 3. Folge, Bd. 33)
Includes bibliographical references and index.
ISBN 3-540-60451-0 (alk. paper)
1. Constructibility (Set theory) 2. Geometry, Algebraic.
I. Bröcker, Ludwig, 1940- . II. Ruiz, Jesús M. III. Title.
IV. Series.
QA248.A67 1996
516'.13--dc20 95-51853
CIP

Mathematics Subject Classification (1991): 12Jxx, 13Fxx, 13Jxx, 14Pxx

ISBN-13: 978-3-642-80026-9 e-ISBN-13: 978-3-642-80024-5
DOI: 10.1007/978-3-642-80024-5

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other ways, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1996
Softcover reprint of the hardcover 1st edition 1996

SPIN 10518835 41/3143 - 5 4 3 2 1 0 – Printed on acid-free paper

Preface

The plan to write this book was laid out in April 1987 at Oberwolfach, during the Conference "Reelle Algebraische Geometrie". Afterwards we met at various conferences and seminars in Luminy, Madrid, Münster, Oberwolfach, Segovia, Soesterberg, Trento and La Turballe. We would like to thank the organizers and the institutions which supported these meetings. With pleasure we remember the special year on Real Algebraic Geometry and Quadratic Forms (Ragsquad) in Berkeley 1990/91 where an essential part of this book was written. Thanks to T.Y. Lam and R. Robson.

We are indepted to our Departments: Universidad Complutense de Madrid and Westfälische Wilhelms-Universität Münster, as well as the D.A.A.D. and the D.G.I.C.Y.T..

It is not possible to mention here all colleagues and friends who showed permanent interest in the project. They encouraged us to continue and complete this work. In particular, we are obliged to Jacek Bochnak, Mike Buchner, Michel Coste and Claus Scheiderer for proofreading and helpful suggestions. While the work was still in progress, Manfred Knebusch used parts of our manuscript for a course on Real Algebraic Geometry. His experience convinced us that it was worth pursuing a fully abstract approach. We were also in permanent contact with Murray Marshall, and the reader will recognize the mutual influence of ideas.

We are also indebted to Professor Reinhold Remmert and to the Springer-Verlag for publishing the book in this series.

Finally, we thank Erika Becker for her care in converting many pages of handwritten mathematics into beautiful TEX.

Madrid, Münster, Majadahonda *C. Andradas, L. Bröcker, J. M. Ruiz*
March 1996

Contents

Introduction 1

Chapter I. A First Look at Semialgebraic Geometry 5

1. Real Closed Fields and Transfer Principles 5
2. What is Semialgebraic Geometry? 9
3. Real Spaces 11
4. Examples 21

Chapter II. Real Algebra 29

1. The Real Spectrum of a Ring 29
2. Specializations, Zero Sets and Real Ideals 35
3. Real Valuations 38
4. Real Going-Up and Real Going-Down 43
5. Abstract Semialgebraic Functions 46
6. Cylindrical Decomposition 51
7. Real Strict Localization 53

Notes 59

Chapter III. Spaces of Signs 61

1. The Axioms 61
2. Forms 68
3. SAP-Spaces and Fans 71
4. Local Spaces of Signs 76
5. The Space of Signs of a Ring 79
6. Subspaces 82

Notes 83

Chapter IV. Spaces of Orderings 85

1. The Axioms Revisited 85
2. Basic Constructions 89

3. Spaces of Finite Type 95
4. Spaces of Finite Chain Length 98
5. Finite Type = Finite Chain Length 105
6. Local-Global Principles 111
7. Representation Theorem and Invariants 115
Notes 122

Chapter V. The Main Results 125

1. Stability Formulae 125
2. Complexity of Constructible Sets 129
3. Separation 134
4. Real Divisors 135
5. The Artin-Lang Property 139
Notes 143

Chapter VI. Spaces of Signs of Rings 145

1. Fans and Valuations 145
2. Field Extensions: Upper Bounds 150
3. Field Extensions: Lower Bounds 154
4. Algebras 158
5. Algebras Finitely Generated over Fields 161
6. Archimedean Rings 164
7. Coming Back to Geometry 168
Notes 176

Chapter VII. Real Algebra of Excellent Rings 177

1. Regular Homomorphisms 177
2. Excellent Rings 182
3. Extension of Orderings Under Completion 187
4. Curve Selection Lemma 189
5. Dimension, Valuations and Fans 195
6. Closures of Constructible Sets 198
7. Real Going-down for Regular Homomorphisms 201
8. Connected Components of Constructible Sets 204
Notes 209

Chapter VIII. Real Analytic Geometry 211

1. Semianalytic Sets 211
2. Semianalytic Set Germs 213
3. Cylindrical Decomposition of Germs 219

4. Rings of Global Analytic Functions 224
5. Hilbert's 17th Problem and Real Nullstellensatz 229
6. Minimal Generation of Global Semianalytic Sets 234
7. Topology of Global Semianalytic Sets 240
8. Germs at Compact Sets 244
Notes 252

Bibliography 255

Glossary 265

Index 267