

Graduate Texts in Mathematics **162**

*Editorial Board*

S. Axler F.W. Gehring P.R. Halmos

**Springer Science+Business Media, LLC**

# Graduate Texts in Mathematics

- 1 TAKEUTI/ZARING. Introduction to Axiomatic Set Theory. 2nd ed.
- 2 OXTOBY. Measure and Category. 2nd ed.
- 3 SCHAEFER. Topological Vector Spaces.
- 4 HILTON/STAMMBACH. A Course in Homological Algebra.
- 5 MAC LANE. Categories for the Working Mathematician.
- 6 HUGHES/PIPER. Projective Planes.
- 7 SERRE. A Course in Arithmetic.
- 8 TAKEUTI/ZARING. Axiomatic Set Theory.
- 9 HUMPHREYS. Introduction to Lie Algebras and Representation Theory.
- 10 COHEN. A Course in Simple Homotopy Theory.
- 11 CONWAY. Functions of One Complex Variable I. 2nd ed.
- 12 BEALS. Advanced Mathematical Analysis.
- 13 ANDERSON/FULLER. Rings and Categories of Modules. 2nd ed.
- 14 GOLUBITSKY/GUILLEMIN. Stable Mappings and Their Singularities.
- 15 BERBERIAN. Lectures in Functional Analysis and Operator Theory.
- 16 WINTER. The Structure of Fields.
- 17 ROSENBLATT. Random Processes. 2nd ed.
- 18 HALMOS. Measure Theory.
- 19 HALMOS. A Hilbert Space Problem Book. 2nd ed.
- 20 HUSEMOLLER. Fibre Bundles. 3rd ed.
- 21 HUMPHREYS. Linear Algebraic Groups.
- 22 BARNES/MACK. An Algebraic Introduction to Mathematical Logic.
- 23 GREUB. Linear Algebra. 4th ed.
- 24 HOLMES. Geometric Functional Analysis and Its Applications.
- 25 HEWITT/STROMBERG. Real and Abstract Analysis.
- 26 MANES. Algebraic Theories.
- 27 KELLEY. General Topology.
- 28 ZARISKI/SAMUEL. Commutative Algebra. Vol. I.
- 29 ZARISKI/SAMUEL. Commutative Algebra. Vol. II.
- 30 JACOBSON. Lectures in Abstract Algebra I. Basic Concepts.
- 31 JACOBSON. Lectures in Abstract Algebra II. Linear Algebra.
- 32 JACOBSON. Lectures in Abstract Algebra III. Theory of Fields and Galois Theory.
- 33 HIRSCH. Differential Topology.
- 34 SPITZER. Principles of Random Walk. 2nd ed.
- 35 WERMER. Banach Algebras and Several Complex Variables. 2nd ed.
- 36 KELLEY/NAMIOKA et al. Linear Topological Spaces.
- 37 MONK. Mathematical Logic.
- 38 GRAUERT/FRITZSCHE. Several Complex Variables.
- 39 ARVESON. An Invitation to  $C^*$ -Algebras.
- 40 KEMENY/SNELL/KNAPP. Denumerable Markov Chains. 2nd ed.
- 41 APOSTOL. Modular Functions and Dirichlet Series in Number Theory. 2nd ed.
- 42 SERRE. Linear Representations of Finite Groups.
- 43 GILLMAN/JERISON. Rings of Continuous Functions.
- 44 KENDIG. Elementary Algebraic Geometry.
- 45 LOÈVE. Probability Theory I. 4th ed.
- 46 LOÈVE. Probability Theory II. 4th ed.
- 47 MOISE. Geometric Topology in Dimensions 2 and 3.
- 48 SACHS/WU. General Relativity for Mathematicians.
- 49 GRUENBERG/WEIR. Linear Geometry. 2nd ed.
- 50 EDWARDS. Fermat's Last Theorem.
- 51 KLINGENBERG. A Course in Differential Geometry.
- 52 HARTSHORNE. Algebraic Geometry.
- 53 MANIN. A Course in Mathematical Logic.
- 54 GRAVER/WATKINS. Combinatorics with Emphasis on the Theory of Graphs.
- 55 BROWN/PEARCY. Introduction to Operator Theory I: Elements of Functional Analysis.
- 56 MASSEY. Algebraic Topology: An Introduction.
- 57 CROWELL/FOX. Introduction to Knot Theory.
- 58 KOBLITZ.  $p$ -adic Numbers,  $p$ -adic Analysis, and Zeta-Functions. 2nd ed.
- 59 LANG. Cyclotomic Fields.
- 60 ARNOLD. Mathematical Methods in Classical Mechanics. 2nd ed.

*continued after index*

J.L. Alperin  
with Rowen B. Bell

# Groups and Representations



Springer

J.L. Alperin  
Rowen B. Bell  
Department of Mathematics  
University of Chicago  
Chicago, IL 60637-1514

*Editorial Board*

S. Axler  
Department of  
Mathematics  
Michigan State University  
East Lansing, MI 48824  
USA

F.W. Gehring  
Department of  
Mathematics  
University of Michigan  
Ann Arbor, MI 48109  
USA

P.R. Halmos  
Department of  
Mathematics  
Santa Clara University  
Santa Clara, CA 95053  
USA

---

Mathematics Subject Classifications (1991): 20-01

---

Library of Congress Cataloging-in-Publication Data

Alperin, J.L.

Groups and representations / J.L. Alperin with Rowen B. Bell.

p. cm. — (Graduate texts in mathematics ; 162)

Includes bibliographical references (p. — ) and index.

ISBN 978-0-387-94526-2 ISBN 978-1-4612-0799-3 (eBook)

DOI 10.1007/978-1-4612-0799-3

1. Representations of groups. I. Bell, Rowen B. II. Title.

III. Series.

QA176.A46 1995

512'.2—dc20

95-17160

Printed on acid-free paper.

© 1995 Springer Science+Business Media New York

Originally published by Springer-Verlag New York, Inc in 1995

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher Springer Science+Business Media, LLC, except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

Production managed by Robert Wexler; manufacturing supervised by Jeffrey Taub.

Photocomposed copy prepared from the author's LaTeX file.

9 8 7 6 5 4 3 2 1

ISBN 978-0-387-94526-2

# Preface

This book is based on a first-year graduate course given regularly by the first author at the University of Chicago, most recently in the autumn quarters of 1991, 1992, and 1993. The lectures given in this course were expanded and prepared for publication by the second author.

The aim of this book is to provide a concise yet thorough treatment of some topics from group theory and representation theory with which every mathematician should be well acquainted. Of course, the topics covered naturally reflect the viewpoints and interests of the authors; for instance, we make no mention of free groups, and the emphasis throughout is admittedly on finite groups. Our hope is that this book will enable graduate students from every mathematical field, as well as bright undergraduates with an interest in algebra, to solidify their knowledge of group theory.

As the course on which this book is based is required for all incoming mathematics graduate students at Chicago, we make very modest assumptions about the algebraic background of the reader. A nodding familiarity with groups, rings, and fields, along with some exposure to elementary number theory and a solid knowledge of linear algebra (including, at times, familiarity with canonical forms of matrices), should be sufficient preparation.

We now give a brief summary of the book's contents. The first four chapters are devoted to group theory. Chapter 1 contains a review (largely without proofs) of the basics of group theory, along with material on automorphism groups, semidirect products, and group actions. These latter concepts are among our primary tools in the book and are often not covered adequately during one's first exposure to group theory. Chapter 2 discusses the structure of the general linear groups and culminates with a proof of the simplicity of the projective special linear groups. An understanding of this material is an essential (but often overlooked) component of any substantive study of group theory; for, as the first author once wrote:

The typical example of a finite group is  $GL(n, q)$ , the general linear group of  $n$  dimensions over the field with  $q$  elements. The student who is introduced to the subject with other examples is being completely misled. [3, p. 121]

Chapter 3 concentrates on the examination of finite groups through their  $p$ -subgroups, beginning with Sylow's theorem and moving on to such results as the Schur-Zassenhaus theorem. Chapter 4 starts with the Jordan-Hölder theorem and continues with a discussion of solvable and nilpotent groups. The final two chapters focus on finite-dimensional algebras and the representation theory of finite groups. Chapter 5 is centered around Maschke's theorem and Wedderburn's structure theorems for semisimple algebras. Chapter 6 develops the ordinary character theory of finite groups, including induced characters, while the Appendix treats some additional topics in character theory that require a somewhat greater algebraic background than does the core of the book.

We have included close to 200 exercises, and they form an integral part of the book. We have divided these problems into "exercises" and "further exercises;" the latter category is generally reserved for exercises that introduce and develop theoretical concepts not included in the text. The level of the problems varies from routine to difficult, and there are a few that we do not expect any student to be able to handle. We give no indication of the degree of difficulty of each exercise, for in mathematical research one does not know in advance what amount of work will be required to complete any step! In an effort to keep our exposition self-contained, we have strived to keep references in the text to the exercises at a minimum.

The sections of this book are numbered continuously, so that Section 4 is actually the first section of Chapter 2, and so forth. A citation of the form “Proposition Y” refers to the result of that name in the current section, while a citation of the form “Proposition X.Y” refers to Proposition Y of Section X.

We would like to extend our thanks to: Michael Maltenfort and Colin Rust, for their thought-provoking proofreading and their many constructive suggestions during the preparation of this book; the students in the first author’s 1993 course, for their input on an earlier draft of this book which was used as that course’s text; Efim Zelmanov and the students in his 1994 Chicago course, for the same reason; and the University of Chicago mathematics department, for continuing to provide summer support for graduate students, as without such support this book would not have been written in its present form. We invite you to send notice of errors, typographical or otherwise, to the second author at [bell@math.uchicago.edu](mailto:bell@math.uchicago.edu).

In remembrance of a life characterized by integrity, devotion to family, and service to community, the second author would like to dedicate this book to David Wellman (1953–1995).

# Contents

<b>Preface</b> .....	<b>v</b>
<b>1. Rudiments of Group Theory</b> .....	<b>1</b>
1. Review .....	1
2. Automorphisms.....	14
3. Group Actions.....	27
<b>2. The General Linear Group</b> .....	<b>39</b>
4. Basic Structure .....	39
5. Parabolic Subgroups .....	49
6. The Special Linear Group.....	56
<b>3. Local Structure</b> .....	<b>63</b>
7. Sylow's Theorem.....	63
8. Finite $p$ -groups .....	72
9. The Schur-Zassenhaus Theorem .....	81



<b>4. Normal Structure</b> .....	<b>89</b>
10. Composition Series .....	89
11. Solvable Groups .....	95
<b>5. Semisimple Algebras</b> .....	<b>107</b>
12. Modules and Representations .....	107
13. Wedderburn Theory .....	120
<b>6. Group Representations</b> .....	<b>137</b>
14. Characters .....	137
15. The Character Table .....	146
16. Induction .....	164
<b>Appendix: Algebraic Integers and Characters</b> .....	<b>179</b>
<b>Bibliography</b> .....	<b>185</b>
<b>List of Notation</b> .....	<b>187</b>
<b>Index</b> .....	<b>191</b>