C H A O S An Introduction to Dynamical Systems

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CHAOS

An Introduction to Dynamical Systems

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Introduction

Background

Sir Isaac Newton brought to the world the idea of modeling the motion of physical systems with equations. It was necessary to invent calculus along the way, since fundamental equations of motion involve velocities and accelerations, which are derivatives of position. His greatest single success was his discovery that the motion of the planets and moons of the solar system resulted from a single fundamental source: the gravitational attraction of the bodies. He demonstrated that the observed motion of the planets could be explained by assuming that there is a gravitational attraction between any two objects, a force that is proportional to the product of masses and inversely proportional to the square of the distance between them. The circular, elliptical, and parabolic orbits of astronomy were no longer fundamental determinants of motion, but were approximations of laws specified with differential equations. His methods are now used in modeling motion and change in all areas of science.

Subsequent generations of scientists extended the method of using differential equations to describe how physical systems evolve. But the method had a limitation. While the differential equations were sufficient to determine the behavior—in the sense that solutions of the equations did exist—it was frequently difficult to figure out what that behavior would be. It was often impossible to write down solutions in relatively simple algebraic expressions using a finite number of terms. Series solutions involving infinite sums often would not converge beyond some finite time.

When solutions could be found, they described very regular motion. Generations of young scientists learned the sciences from textbooks filled with examples of differential equations with regular solutions. If the solutions remained in a bounded region of space, they settled down to either (A) a steady state, often due to energy loss by friction, or (B) an oscillation that was either periodic or quasiperiodic, akin to the clocklike motion of the moon and planets. (In the solar system, there were obviously many different periods. The moon traveled around the earth in a month, the earth around the sun in about a year, and Jupiter around the sun in about 11.867 years. Such systems with multiple incommensurable periods came to be called quasiperiodic.)

Scientists knew of systems which had more complicated behavior, such as a pot of boiling water, or the molecules of air colliding in a room. However, since these systems were composed of an immense number of interacting particles, the complexity of their motions was not held to be surprising.

Around 1975, after three centuries of study, scientists in large numbers around the world suddenly became aware that there is a third kind of motion, a type (C) motion, that we now call "chaos". The new motion is erratic, but not simply quasiperiodic with a large number of periods, and not necessarily due to a large number of interacting particles. It is a type of behavior that is possible in very simple systems.

A small number of mathematicians and physicists were familiar with the existence of a third type of motion prior to this time. James Clerk Maxwell, who studied the motion of gas molecules in about 1860, was probably aware that even a system composed of two colliding gas particles in a box would have neither motion type A nor B, and that the long term behavior of the motions would for all practical purposes be unpredictable. He was aware that very small changes in the initial motion of the particles would result in immense changes in the trajectories of the molecules, even if they were thought of as hard spheres.

Maxwell began his famous study of gas laws by investigating individual collisions. Consider two atoms of equal mass, modeled as hard spheres. Give the atoms equal but opposite velocities, and assume that their positions are selected at random in a large three-dimensional region of space. Maxwell showed that if they collide, all directions of travel will be equally likely after the collision. He recognized that small changes in initial positions can result in large changes in outcomes. In a discussion of free will, he suggested that it would be impossible to test whether a leopard has free will, because one could never compute from a study of its atoms what the leopard would do. But the chaos of its atoms is limited, for, as he observed, "No leopard can change its spots!"

Henri Poincaré in 1890 studied highly simplified solar systems of three bodies and concluded that the motions were sometimes incredibly complicated. (See Chapter 2). His techniques were applicable to a wide variety of physical systems. Important further contributions were made by Birkhoff, Cartwright and Littlewood, Levinson, Kolmogorov and his students, among others. By the 1960s, there were groups of mathematicians, particularly in Berkeley and in Moscow, striving to understand this third kind of motion that we now call chaos. But only with the advent of personal computers, with screens capable of displaying graphics, have scientists and engineers been able to see that important equations in their own specialties had such solutions, at least for some ranges of parameters that appear in the equations.

In the present day, scientists realize that chaotic behavior can be observed in experiments and in computer models of behavior from all fields of science. The key requirement is that the system involve a nonlinearity. It is now common for experiments whose previous anomalous behavior was attributed to experiment error or noise to be reevaluated for an explanation in these new terms. Taken together, these new terms form a set of unifying principles, often called dynamical systems theory, that cross many disciplinary boundaries.

The theory of dynamical systems describes phenomena that are common to physical and biological systems throughout science. It has benefited greatly from the collision of ideas from mathematics and these sciences. The goal of scientists and applied mathematicians is to find nature's unifying ideas or laws and to fashion a language to describe these ideas. It is critical to the advancement of science that exacting standards are applied to what is meant by knowledge. Beautiful theories can be appreciated for their own sake, but science is a severe taskmaster. Intriguing ideas are often rejected or ignored because they do not meet the standards of what is knowledge.

The standards of mathematicians and scientists are rather different. Mathematicians prove theorems. Scientists look at realistic models. Their approaches are somewhat incompatible. The first papers showing chaotic behavior in computer studies of very simple models were distasteful to both groups. The mathematicians feared that nothing was proved so nothing was learned. Scientists said that models without physical quantities like charge, mass, energy, or acceleration could not be relevant to physical studies. But further reflection led to a change in viewpoints. Mathematicians found that these computer studies could lead to new ideas that slowly yielded new theorems. Scientists found that computer studies of much more complicated models yielded behaviors similar to those of the simplistic models, and that perhaps the simpler models captured the key phenomena.

Finally, laboratory experiments began to be carried out that showed unequivocal evidence of unusual nonlinear effects and chaotic behavior in very familiar settings. The new dynamical systems concepts showed up in macroscopic systems such as fluids, common electronic circuits and low-energy lasers that were previously thought to be fairly well understood using the classical paradigms. In this sense, the chaotic revolution is quite different than that of relativity, which shows its effects at high energies and velocities, and quantum theory, whose effects are submicroscopic. Many demonstrations of chaotic behavior in experiments are not far from the reader's experience.

In this book we study this field that is the uncomfortable interface between mathematics and science. We will look at many pictures produced by computers and we try to make mathematical sense of them. For example, a computer study of the driven pendulum in Chapter 2 reveals irregular, persistent, complex behavior for ten million oscillations. Does this behavior persist for one billion oscillations? The only way we can find out is to continue the computer study longer. However, even if it continues its complex behavior throughout our computer study, we cannot guarantee it would persist forever. Perhaps it stops abruptly after one trillion oscillations; we do not know for certain. We can prove that there exist initial positions and velocities of the pendulum that yield complex behavior forever, but these choices are conceivably quite atypical. There are even simpler models where we know that such chaotic behavior does persist forever. In this world, pictures with uncertain messages remain the medium of inspiration.

There is a philosophy of modeling in which we study idealized systems that have properties that can be closely approximated by physical systems. The experimentalist takes the view that only quantities that can be measured have meaning. Yet we can prove that there are beautiful structures that are so infinitely intricate that they can never be seen experimentally. For example, we will see immediately in Chapters 1 and 2 the way chaos develops as a physical parameter like friction is varied. We see infinitely many periodic attractors appearing with infinitely many periods. This topic is revisited in Chapter 12, where we show

how this rich bifurcation structure, called a cascade, exists with mathematical certainty in many systems. This is a mathematical reality that underlies what the experimentalist can see. We know that as the scientist finds ways to make the study of a physical system increasingly tractable, more of this mathematical structure will be revealed. It is there, but often hidden from view by the noise of the universe. All science is of course dependent on simplistic models. If we study a vibrating beam, we will generally not model the atoms of which it is made. If we model the atoms, we will probably not reflect in our model the fact that the universe has a finite age and that the beam did not exist for all time. And we do not include in our model (usually) the tidal effects of the stars and the planets on our vibrating beam. We ignore all these effects so that we can isolate the implications of a very limited list of concepts.

It is our goal to give an introduction to some of the most intriguing ideas in dynamics, the ideas we love most. Just as chemistry has its elements and physics has its elementary particles, dynamics has its fundamental elements: with names like attractors, basins, saddles, homoclinic points, cascades, and horseshoes. The ideas in this field are not transparent. As a reader, your ability to work with these ideas will come from your own effort. We will consider our job to be accomplished if we can help you learn what to look for in your own studies of dynamical systems of the world and universe.

ABOUT THE BOOK

As we developed the drafts of this book, we taught six one semester classes at George Mason University and the University of Maryland. The level is aimed at undergraduates and beginning graduate students. Typically, we have used parts of Chapters 1–9 as the core of such a course, spending roughly equal amounts of time on iterated maps (Chapters 1–6) and differential equations (Chapters 7–9). Some of the maps we use as examples in the early chapters come from differential equations, so that their importance in the subject is stressed. The topics of stable manifolds, bifurcations, and cascades are introduced in the first two chapters and then developed more fully in the Chapters 10, 11, and 12, respectively. Chapter 13 on time series may be profitably read immediately after Chapter 4 on fractals, although the concepts of periodic orbit (of a differential equation) and chaotic attractor will not yet have been formally defined.

The impetus for advances in dynamical systems has come from many sources: mathematics, theoretical science, computer simulation, and experimental science. We have tried to put this book together in a way that would reflect its wide range of influences.

We present elaborate dissections of the proofs of three deep and important theorems: The Poincaré-Bendixson Theorem, the Stable Manifold Theorem, and the Cascade Theorem. Our hope is that including them in this form tempts you to work through the nitty-gritty details, toward mastery of the building blocks as well as an appreciation of the completed edifice.

Additionally, each chapter contains a special feature called a Challenge, in which other famous ideas from dynamics have been divided into a number of steps with helpful hints. The Challenges tackle subjects from period-three implies chaos, the cat map, and Sharkovskii's ordering through synchronization and renormalization. We apologize in advance for the hints we have given, when they are of no help or even mislead you; for one person's hint can be another's distraction.

The Computer Experiments are designed to present you with opportunities to explore dynamics through computer simulation, the venue through which many of these concepts were first discovered. In each, you are asked to design and carry out a calculation relevant to an aspect of the dynamics. Virtually all can be successfully approached with a minimal knowledge of some scientific programming language. Appendix B provides an introduction to the solution of differential equations by approximate means, which is necessary for some of the later Computer Experiments.

If you prefer not to work the Computer Experiments from scratch, your task can be greatly simplified by using existing software. Several packages are available. *Dynamics: Numerical Explorations* by H.E. Nusse and J.A. Yorke (Springer-Verlag 1994) is the result of programs developed at the University of Maryland. *Dynamics*, which includes software for Unix and PC environments, was used to make many of the pictures in this book. The web site for *Dynamics* is `www.ipst.umd.edu/dynamics`. We can also recommend *Differential and Difference Equations through Computer Experiments* by H. Kocak (Springer-Verlag, 1989) for personal computers. A sophisticated package designed for Unix platforms is *dstool*, developed by J. Guckenheimer and his group at Cornell University. In the absence of special purpose software, general purpose scientific computing environments such as Matlab, Maple, and Mathematica will do nicely.

The Lab Visits are short reports on carefully selected laboratory experiments that show how the mathematical concepts of dynamical systems manifest themselves in real phenomena. We try to impart some flavor of the setting of the experiment and the considerable expertise and care necessary to tease a new secret from nature. In virtually every case, the experimenters' findings far surpassed what we survey in the Lab Visit. We urge you to pursue more accurate and detailed discussions of these experiments by going straight to the original sources.

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