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Topology of Real Algebraic Sets



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Selman Akbulut
Department of Mathematics
Michigan State University
East Lansing, MI 48824
USA

Henry King
Department of Mathematics
University of Maryland
College Park, MD 20742
USA

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Preface

In the Fall of 1975 we started a joint project with the ultimate goal of topologically classifying real algebraic sets. This has been a long happy collaboration (c.f., [K2]). In 1985 while visiting M.S.R.I. we organized and presented our classification results up to that point in the M.S.R.I. preprint series [AK14]-[AK17]. Since these results are interdependent and require some prerequisites as well as familiarity with real algebraic geometry, we decided to make them self contained by presenting them as a part of a book in real algebraic geometry. Even though we have not arrived to our final goal yet we feel that it is time to introduce them in a self contained coherent version and demonstrate their use by giving some applications.

Chapter I gives the overview of the classification program. Chapter II has all the necessary background for the rest of the book, which therefore can be used as a course in real algebraic geometry. It starts with the elementary properties of real algebraic sets and ends with the recent solution of the Nash Conjecture. Chapter III and Chapter IV develop the theory of resolution towers. Resolution towers are basic topologically defined objects generalizing the notion of manifold. They enable us to study singular spaces in an organized way. Chapter V shows how to obtain algebraic sets from resolution towers. Chapter VI shows how to put resolution tower structures on real or complex algebraic sets. Chapter VII applies this theory to real algebraic sets of dimension ≤ 3 by giving their topological characterization. An impatient reader can go directly to Chapter VII from Chapter I in order to get motivated for the results of Chapter III through Chapter VI .

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We now fix some notation, some of it nonstandard, which we will use throughout the book. We let \mathbf{R} and \mathbf{C} denote the real and complex numbers. We let I denote the closed interval $[0, 1]$ in \mathbf{R} . If A is a subset of a topological space then we let $\text{Cl}(A)$ denote the closure of A and let $\text{Int}(A)$ denote its interior. If A and B are sets, then $A - B$ denotes their difference. If $f: M \rightarrow N$ is a smooth map between smooth manifolds, we let $df: TM \rightarrow TN$ denote the induced mapping on tangent spaces. The expression $A \sqcup B$ means the disjoint union. A closed manifold means a compact manifold without boundary.

We now introduce some nonstandard notation. If $f: M \rightarrow N$ is a function and $S \subset M$ we will let $f|_S$ denote the restriction $f|_S$ if S is clear from context. This is useful if S is some complicated expression which would only clutter up a formula and make it more unreadable. If X is a topological space, we let $\mathbf{c}X$ denote the cone on X , so $\mathbf{c}X = X \times [0, 1]/X \times 0$, the quotient space of $X \times [0, 1]$ with $X \times 0$ crushed to a point. If x (or y or z etc.) is a point in \mathbf{R}^n then x_i (or y_i or z_i) will denote the i -th coordinate of x . We let \mathbf{R}_i^n denote the coordinate hyperplane $\{x \in \mathbf{R}^n \mid x_i = 0\}$.

The end of a proof is marked by a sign thusly: □

We have tried to organize long proofs in a hierarchical manner. In the midst of a proof we may make an assertion, which we then proceed to prove. The reader might prefer to skip this proof on first reading or do it as an exercise if she is energetic. The hope is that this will make the overall argument clearer by hiding some of the details. To set off the proof of an assertion from the rest of the proof we mark its end thusly: □

Contents

Preface	v
List of Figures	ix
Chapter I. INTRODUCTION	1
1. Overview	1
2. Stratified Sets	5
3. Ticos	6
4. Resolution Towers	7
Chapter II. ALGEBRAIC SETS	17
1. Basic Properties of Algebraic Sets	17
2. Singularities of Real and Complex Algebraic Sets	22
3. Projective Algebraic Sets	33
4. Grassmannians	37
5. Blowing Up	40
6. Blowing Down	50
7. Algebraic Homology	53
8. Making Smooth Objects Algebraic	57
9. Homology of Blowups	77
10. Isotoping Submanifolds to Algebraic Subsets	85
Chapter III. TICOS	93
1. Some Results about Smooth Functions	93
2. Ticos	95

3. Tico Blowups	104
4. Full Ticos	120
5. Type N Tico Maps	121
6. Submersive Tico Maps	131
7. Micos	133
Chapter IV. RESOLUTION TOWERS	136
1. Definiton of Resolution Towers	136
2. Blowing up Resolution Towers	141
3. Realizations of Resolution Towers	151
Chapter V. ALGEBRAIC STRUCTURES ON RESOLUTION TOWERS	158
1. Making Tico Maps Algebraic	158
2. Nice Charts on Resolution Towers	161
3. Quasialgebraic Towers are Algebraic	167
4. RF Towers are Quasialgebraic	170
Chapter VI. RESOLUTION TOWER STRUCTURES ON ALGEBRAIC SETS	173
1. Uzunblowups and Fullness	174
2. Complex Ticos and Complexifications	176
3. Extending Algebraic Resolution Towers	183
4. Resolution Towers for Algebraic Sets	186
Chapter VII. THE CHARACTERIZATION OF THREE DIMENSIONAL ALGEBRAIC SETS	191
1. Obstructions	193
2. The Cobordism Groups	202
3. Characterization in Dimension 3	213
4. Algebraic Resolution of Real Algebraic Sets in Dimension Three	219
5. Bounding Resolution Towers	225
Bibliography	244
Index	247

List of Figures

I.1.1	A noncompact real algebraic set	2
I.1.2	A_1 spaces with and without boundary	2
I.1.3	An algebraic set which is not an A_k -space	3
I.1.4	A topological resolution	3
I.1.5	A resolution tower for Z	4
I.4.1	A resolution tower	8
I.4.2	Realization of a resolution tower	9
I.4.3	X	10
I.4.4	Tangential intersection	11
I.4.5	$\pi^{-1}(X_1)$ pairwise transverse, but not a tico	12
I.4.6	$\pi^{-1}(X_1)$ now a tico	12
I.4.7	$\pi^{-1}(X_0)$ now a tico	13
II.4.1	Projecting to the normal bundle of L in V	39
II.6.1	Algebraic blowing down	51
II.6.2	Squishing a sphere to a figure 8	52
II.6.3	Squishing a hyperboloid to a parabola	53
II.7.1	V and Y_j	56
II.8.1	Making a map algebraic	63
II.8.2	The doubled cobordism	66
II.8.3	The doubled cobordism made algebraic	66
II.8.4	Making an algebraic embedding into an algebraic subset	69
II.8.5	A smooth approximation	69
II.8.6	The cobordism to the algebraic situation	72
II.8.7	The inductive step, reducing the number of nonempty M_α 's	73
II.8.8	Detail of the added handle	73
II.8.9	X , the double of T	74
II.8.10	Y_i''	75

II.8.11	Characterizing Zopen sets with isolated singularities	76
II.8.12	Balls D_i making a spine of W	76
II.8.13	$\bigcup S_i$ a spine of W_0	77
II.8.14	Adding a handle to reduce the number of spheres in ∂W_i	77
II.10.1	An almost nonsingular immersion	88
II.10.2	Making an immersion algebraic	90
III.2.1	A tico in the disc	96
IV.1.1	A realization of a resolution tower	138
IV.1.2	A resolution tower	138
IV.1.3	The realization of the resolution tower	138
VII.1.1	Y_0	195
VII.1.2	Y_1	196
VII.1.3	Y_2	196
VII.1.4	Y_3	197
VII.2.1	\mathfrak{T}' and its resolution	205
VII.2.2	A typical tower in \mathcal{TR}_2	206
VII.2.3	Some generators of \mathfrak{N}_2^S	212
VII.3.1	X and a resolution \mathfrak{T} of $X - K$	215
VII.3.2	The link L and its induced resolution tower \mathfrak{T}'''	215
VII.3.3	A resolution tower \mathfrak{T}'' for cL	216
VII.3.4	A resolution tower \mathfrak{T}' for X	216
VII.5.1	Pairing up points of V_1 and V_{12}	226
VII.5.2	The realization of \mathfrak{T}	226
VII.5.3	Starting to make \mathfrak{T} a weak boundary	227
VII.5.4	Handles added until each $T \cap C_{02}$ has at most one point	228
VII.5.5	Handles added until each $T \cap C_{02}$ has one point	228
VII.5.6	Adding a round handle	228
VII.5.7	Resolving the cone on the 0-skeleton	230
VII.5.8	The first two pieces of V_2'	232
VII.5.9	The first pieces of V_3'	233
VII.5.10	Covering a new handle in C_2 by one in C_3	234
VII.5.11	Another type of handle we may add to C_2	235
VII.5.12	The involution τ	236
VII.5.13	A handle pair added to C_2	237
VII.5.14	Adding a handle to C_3 which reduces (d, e, f)	238
VII.5.15	Covering with a round handle	239