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Volume 25	Akbulut and King: Topology of Real Algebraic Sets

Selman Akbulut Henry King

Topology of Real Algebraic Sets

Springer-Verlag
New York Berlin Heidelberg London Paris
Tokyo Hong Kong Barcelona Budapest

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The Mathematical Sciences Research Institute wishes to acknowledge the support by the National Sciences Foundation.

Mathematical Subject Classifications: 14P25, 57N80, 32C05, 58A35

Library of Congress Cataloging-in-Publication Data
Akbulut, Selman, 1948-
Topology of real algebraic sets / Selman Akbulut, Henry King.
p. cm -- (Mathematical Sciences Research Institute publications : 25)
Includes bibliographical references and index.
ISBN-13:978-1-4613-9741-0 e-ISBN-13:978-1-4613-9739-7
DOI: 10.1007/978-1-4613-9739-7

1. Ordered fields. 2. Geometry, Algebraic. I. King, Henry, 1948- . II. Title. III. Series.
QA247.A42 1992
516.3'5--dc20 91-37834

Printed on acid-free paper.

Production managed by Karen Phillips, Manufacturing supervised by Robert Paella.
Camera-ready copy prepared by the Mathematical Sciences Research Institute using $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX.

9 8 7 6 5 4 3 2 1

ISBN-13:978-1-4613-9741-0

Preface

In the Fall of 1975 we started a joint project with the ultimate goal of topologically classifying real algebraic sets. This has been a long happy collaboration (c.f., **[K2]**). In 1985 while visiting M.S.R.I. we organized and presented our classification results up to that point in the M.S.R.I. preprint series **[AK14]** -**[AK17]**. Since these results are interdependent and require some prerequisites as well as familiarity with real algebraic geometry, we decided to make them self contained by presenting them as a part of a book in real algebraic geometry. Even though we have not arrived to our final goal yet we feel that it is time to introduce them in a self contained coherent version and demonstrate their use by giving some applications.

Chapter I gives the overview of the classification program. Chapter II has all the necessary background for the rest of the book, which therefore can be used as a course in real algebraic geometry. It starts with the elementary properties of real algebraic sets and ends with the recent solution of the Nash Conjecture. Chapter III and Chapter IV develop the theory of resolution towers. Resolution towers are basic topologically defined objects generalizing the notion of manifold. They enable us to study singular spaces in an organized way. Chapter V shows how to obtain algebraic sets from resolution towers. Chapter VI shows how to put resolution tower structures on real or complex algebraic sets. Chapter VII applies this theory to real algebraic sets of dimension ≤ 3 by giving their topological characterization. An impatient reader can go directly to Chapter VII from Chapter I in order to get motivated for the results of Chapter III through Chapter VI .

We would like to thank National Science Foundation, the Institute for Advanced Study, the Max-Planck Institute, the Mathematical Sciences Research Institute, the General Research Board of the University of Maryland as well as our respective universities: Michigan State University and University of Maryland for generous support while this work has been in progress. Also we would like to thank Lowell Jones and Elmer Rees for timely advice. The first named author would like thank his advisor R.Kirby for introducing him to the subject, and his teachers: Fahrettin Akbulut, Írfan Barış, S.S.Chern, Tom Farrell, Moe Hirsch, Dennis Sullivan, Larry Taylor for inspiration, and TUBITAK (Turkish scientific research institute) for the initial support. The second named author would also like to thank Dick Palais for teaching him much about real algebraic geometry and Dennis Sullivan for general mathematical stimulation. We would like to thank J. Bochnak and W. Kucharz for their helpful comments on preliminary versions of this book. We thank D. Glaubman for helping us with some of the computer generated figures. We would like to thank Tammy Hatfield, Cindy Smith, and Cathy Friess for doing a great job of typesetting this book in LaTeX. Finally we would like to thank Margaret Pattison for her help in preparing this book for publication.

We now fix some notation, some of it nonstandard, which we will use throughout the book. We let $\mathbf{R}$ and $\mathbf{C}$ denote the real and complex numbers. We let I denote the closed interval $[0,1]$ in $\mathbf{R}$. If A is a subset of a topological space then we let $\mathrm{Cl}(A)$ denote the closure of A and let $\mathrm{Int}(A)$ denote its interior. If A and B are sets, then $A - B$ denotes their difference. If $f\colon M \to N$ is a smooth map between smooth manifolds, we let $df\colon TM \to TN$ denote the induced mapping on tangent spaces. The expression $A \sqcup B$ means the disjoint union. A closed manifold means a compact manifold without boundary.

We now introduce some nonstandard notation. If $f\colon M \to N$ is a function and $S \subset M$ we will let $f|$ denote the restriction $f|_S$ if S is clear from context. This is useful if S is some complicated expression which would only clutter up a formula and make it more unreadable. If X is a topological space, we let $\mathfrak{c}X$ denote the cone on X, so $\mathfrak{c}X = X \times [0,1]/X \times 0$, the quotient space of $X \times [0,1]$ with $X \times 0$ crushed to a point. If x (or y or z etc.) is a point in $\mathbf{R}^n$ then x_i (or y_i or z_i) will denote the i-th coordinate of x. We let $\mathbf{R}^n_i$ denote the coordinate hyperplane $\{\, x \in \mathbf{R}^n \mid x_i = 0 \,\}$.

The end of a proof is marked by a sign thusly: □

We have tried to organize long proofs in a hierarchical manner. In the midst of a proof we may make an assertion, which we then proceed to prove. The reader might prefer to skip this proof on first reading or do it as an exercise if she is energetic. The hope is that this will make the overall argument clearer by hiding some of the details. To set off the proof of an assertion from the rest of the proof we mark its end thusly: ▫

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