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Hiroaki Aikawa Matts Essén

Potential Theory – Selected Topics



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Authors

Hiroaki Aikawwa
Department of Mathematics
Shimane University
Matsue 690, Japan
E-mail: haikawa@riko.shimane-u.ac.jp

Matts Essén
Department of Mathematics
Uppsala University
Box 480
75106 Uppsala, Sweden
E-mail: matts.essen@math.uu.se

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During the academic years 1992–1994, there was a lot of activity on potential theory at the Department of Mathematics at Uppsala University. The main series of lectures were as follows:

- A An introduction to potential theory and a survey of minimal thinness and rarefiedness. (M. Essén)
- B Potential theory. (H. Aikawa)
- C Analytic capacity. (V. Eiderman)
- D Lectures on a paper of L.-I. Hedberg [24]. (M. Essén)
- E Harmonic measures on fractals (A. Volberg)

These lecture notes contain the lecture series A,B and references for C. The E lectures will appear as department report UUDM 1994:32: Zoltan Balogh, Irina Popovici and Alexander Volberg, Conformally maximal polynomial-like dynamics and invariant harmonic measure (to appear, Ergodic Theory and Dynamical Systems).

H. Aikawa spent the Spring semester 1993 in Uppsala. V. Eiderman spent the Spring semesters 1993 and 1994 here. A. Volberg was in Uppsala during May 1994. In addition to giving excellent series of lectures, our visitors were also very active participants in the mathematical life of the department.

Uppsala September 21, 1994

Matts Essén

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