

R. Abraham  
J.E. Marsden  
T. Ratiu

# Manifolds, Tensor Analysis, and Applications

Second Edition



Springer

Ralph Abraham  
Department of Mathematics  
University of California—  
Santa Cruz  
Santa Cruz, CA 95064  
USA

Jerrold E. Marsden  
Control and Dynamical  
Systems, 107-81  
California Institute of  
Technology  
Pasadena, CA 91125  
USA

Tudor Ratiu  
Department of Mathematics  
University of California—  
Santa Cruz  
Santa Cruz, CA 95064  
USA

*Editors:*

J.E. Marsden  
Control and Dynamical Systems, 107-81  
California Institute of Technology  
Pasadena, CA 91125  
USA

L. Sirovich  
Division of Applied Mathematics  
Brown University  
Providence, RI 02912  
USA

---

Mathematics Subject Classifications (1991): 34-01, 58-01, 70-01, 76-01, 93-01

---

Library of Congress Cataloging-in-Publication Data

Abraham, Ralph

Manifolds, tensor analysis, and applications, Second Edition

(Applied Mathematical Sciences; v. 75)

Bibliography: p. 631

Includes index.

1. Global analysis (Mathematics) 2. Manifolds (Mathematics) 3. Calculus of tensors.

I. Marsden, Jerrold E. II. Ratiu, Tudor S. III. Title. IV. Series.

QA614.A28 1983514.382-1737

ISBN 978-1-4612-6990-8 ISBN 978-1-4612-1029-0 (eBook)

DOI 10.1007/978-1-4612-1029-0

First edition published by Addison-Wesley Publishing Company © 1983

© 1988 by Springer Science+Business Media New York

Originally published by Springer-Verlag, New York Inc. in 1988

Softcover reprint of the hardcover 2nd edition 1988

All rights reserved. No part of this book may be translated or reproduced in any form without written permission from Springer Science+Business Media, LLC.

9 8 7 6 5 4

ISBN 978-1-4612-6990-8

# Contents

<b>Preface</b>	v
<b>Background Notation</b>	vii
<b>CHAPTER 1</b>	
<b>Topology</b>	1
1.1 Topological Spaces	2
1.2 Metric Spaces	9
1.3 Continuity	14
1.4 Subspaces, Products, and Quotients	18
1.5 Compactness	24
1.6 Connectedness	31
1.7 Baire Spaces	37
<b>CHAPTER 2</b>	
<b>Banach Spaces and Differential Calculus</b>	40
2.1 Banach Spaces	40
2.2 Linear and Multilinear Mappings	56
2.3 The Derivative	75
2.4 Properties of the Derivative	83
2.5 The Inverse and Implicit Function Theorems	116
<b>CHAPTER 3</b>	
<b>Manifolds and Vector Bundles</b>	141
3.1 Manifolds	141
3.2 Submanifolds, Products, and Mappings	150
3.3 The Tangent Bundle	157
3.4 Vector Bundles	167
3.5 Submersions, Immersions and Transversality	196
<b>CHAPTER 4</b>	
<b>Vector Fields and Dynamical Systems</b>	238
4.1 Vector Fields and Flows	238
4.2 Vector Fields as Differential Operators	265
4.3 An Introduction to Dynamical Systems	298
4.4 Frobenius' Theorem and Foliations	326
<b>CHAPTER 5</b>	
<b>Tensors</b>	338
5.1 Tensors in Linear Spaces	338
5.2 Tensor Bundles and Tensor Fields	349
5.3 The Lie Derivative: Algebraic Approach	359
5.4 The Lie Derivative: Dynamic Approach	370
5.5 Partitions of Unity	377

<b>CHAPTER 6</b>	
<b>Differential Forms</b>	392
6.1 Exterior Algebra	392
6.2 Determinants, Volumes, and the Hodge Star Operator	402
6.3 Differential Forms	417
6.4 The Exterior Derivative, Interior Product, and Lie Derivative	423
6.5 Orientation, Volume Elements, and the Codifferential	450
<b>CHAPTER 7</b>	
<b>Integration on Manifolds</b>	464
7.1 The Definition of the Integral	464
7.2 Stokes' Theorem	476
7.3 The Classical Theorems of Green, Gauss, and Stokes	504
7.4 Induced Flows on Function Spaces and Ergodicity	513
7.5 Introduction to Hodge-deRham Theory and Topological Applications of Differential Forms	538
<b>CHAPTER 8</b>	
<b>Applications</b>	560
8.1 Hamiltonian Mechanics	560
8.2 Fluid Mechanics	584
8.3 Electromagnetism	599
8.3 The Lie-Poisson Bracket in Continuum Mechanics and Plasma Physics	609
8.4 Constraints and Control	624
<b>References</b>	631
<b>Index</b>	643
<b>Supplementary Chapters</b> —Available from the authors as they are produced	
<b>S-1 Lie Groups</b>	
<b>S-2 Introduction to Differential Topology</b>	
<b>S-3 Topics in Riemannian Geometry</b>	

## *Background Notation*

The reader is assumed to be familiar with the usual notations of set theory such as  $\in$ ,  $\subset$ ,  $\cup$ ,  $\cap$  and with the concept of a mapping. If  $A$  and  $B$  are sets and if  $f: A \rightarrow B$  is a mapping, we write  $a \mapsto f(a)$  for the effect of the mapping on the element of  $a \in A$ ; ‘‘iff’’ stands for ‘‘if and only if’’ (= ‘‘if’’ in definitions). Other notations we shall use without explanation include the following:

◆	end of an example or remark
■	end of a proof
▼	proof of a lemma is done, but the proof of the theorem goes on
$\mathbb{R}, \mathbb{C}$	real, complex numbers
$\mathbb{Z}, \mathbb{Q}$	integers, rational numbers
$A \times B$	Cartesian product
$\mathbb{R}^n, \mathbb{C}^n$	Euclidean $n$ -space, complex $n$ -space
$(x^1, \dots, x^n) \in \mathbb{R}^n$	point in $\mathbb{R}^n$
$A \subset B$	set theoretic containment (means same as $A \subseteq B$ )
$A \setminus B$	set theoretic difference
I or Id	identity map
$f^{-1}(B)$	inverse image of $B$ under $f$
$\Gamma_f = \{(x, f(x)) \mid x \in \text{domain of } f\}$	graph of $f$
$\inf A$	infimum (greatest lower bound) of the set $A \subset \mathbb{R}$
$\sup A$	supremum (least upper bound) of $A \subset \mathbb{R}$
$e_1, \dots, e_n$	basis of an $n$ -dimensional vector space
$\ker T, \text{range } T$	kernel and range of a linear transformation $T$
$D_r(m)$	open ball about $m$ of radius $r$
$B_r(m)$	closed ball of radius $r$ (also denoted $\overline{D}_r(m)$ ).