Kenneth Ireland Michael Rosen

## A Classical Introduction to Modern Number Theory

Second Edition



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