Virtual Element Methods in Engineering Sciences

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Preface

The idea to write this book, related to engineering applications of the new virtual element method, occurred to us when the lock-down during the Coronavirus pandemic started. There was suddenly time to concentrate at home on an aggregation of work that we had done together with my group and colleagues. It all started in 2016 after having heard several inspiring talks by Franco Brezzi, his wife Donatella Marini and Lourenco Beirão da Veiga on the new virtual element method. It was immediately clear that this methodology might have advantages for numerical solution schemes in different applications, especially in the nonlinear range.

In 2016, we had first discussions on the basis of the virtual element method and how to implement it with Daya Reddy who stayed at the institute with an award from the Alexander von Humboldt Foundation. The idea was to use virtual elements for contact with the advantage that even for non-matching meshes a node to node formulation of contact was possible. Together with Wilhelm Rust we derived a contact discretization and algorithm based on virtual elements, first for frictionless and later for frictional contact in large strain applications. A work to be continued with our colleagues from Italy, Lourenco Beirão da Veiga and Edoardo Artioli for curved virtual elements. A contribution to three-dimensional contact is due to Mertcan Cihan who has developed in his dissertation the complex projection procedures needed in contact discretizations for three-dimensions when using virtual elements. Also Alfredo Gay Neto from the University of Sao Paulo, a former Humboldt fellow in our institute, contributed to contact formulations within the virtual element method by integrating three-dimensional virtual elements for finite elastic strains as single flexible particles into his discrete element code.

With Daya we tried to understand finite strain problems in 2017 and got some new ideas about the stabilization of the method for low order approximations. After that we worked on a quite general implementation that could be applied using two- and three-dimensional meshes with virtual elements of arbitrary shape. With these tools at hand it was only a small step to apply the new virtual element scheme also to finite strain plasticity problems and extend the application range to anisotropic materials which resulted in joint work with Jörg Schröder. The virtual element method was further extended to large strain dynamical problems by Mertcan Cihan who applied his developments to vibration problems in elasticity and impact problems undergoing finite elasto-plastic deformations.

Fracture mechanics is another application where the virtual element method offers some advantages. To cover different possible approaches, we tried the phase field method for brittle and ductile fracture with Fadi Aldakheel. But also a new cutting scheme, based on linear fracture mechanics, was introduced within the virtual element method in the Ph.D. thesis of Ali Hussein. Additionally, phase field (for fracture detection) and the cutting scheme were put together in this thesis with adaptivity and provided a new efficient tool for crack propagation problems.

When Maria Laura de Bellis came to our institute as a Humboldt fellow, we continued our work on the virtual element method in the area of damage mechanics. Here virtual elements were used in a non-local form. Additionally, Laura worked on a serendipity formulation of virtual elements for finite strain problems.

With another Humboldian, Michele Marino, we tackled homogenization problems where the vitual element method has big advantages due to the possibility to define elements with arbitrary shape. Here, metals and ceramics are described using a direct discretization of a real microstructure by one virtual element per grains with non-convex polyhedral shape. This provides a very efficient tool for homogenization, especially since only averages have to be computed. This is also demonstrated in the Ph.D. work by Christoph Böhm for crystalline microstructures of steel and magnetoelectro-mechanical materials.

Lately, we applied the virtual element method to Kirchhoff plates where C^1-continuous ansatz functions are needed. Based on the pioneering work of Franco Brezzi and Donatella Marini we were able to construct together with Olivier Allix, who visited our institute as a Humboldt laureate, virtual plate elements that provide stable solutions for thin plates with isotropic and anisotropic material.

This book is a combined effort of the three of us to allow graduate students and engineers working in industry to familiarize themselves with the new virtual element method. Due to that examples are included in the text for deeper understanding of the formulations and associated algorithms. Additionally, a simplified code that allows to solve two-dimensional elastic problems with the software tools *AceGen* and *AceFEM* was created and can be found in the shared library *AceShare* of *AceGen*.

We are very grateful to all Ph.D. students, post-docs and colleagues mentioned above with whom we had extremely fruitful discussions and collaborations. But we also have to mention and thank Jože Korelc, Professor in Ljubljana, who provided his symbolic software tool *AceGen* and the related analysis code *AceFEM* and changed it during the last five years considerably in order to accommodate the need for more flexibility when using a discretization scheme like the virtual element method with arbitrary and different number of nodes within elements and meshes.

Last but not least, we like to thank Volker Meine for drawing many of the figures in the book.

Hannover, Germany
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